Mathematical Meanings for Teaching secondary mathematics

MMTsm

Form 12

A Project of the Arizona State University and UC Berkeley[1]

Dear Mathematics Teacher:

Thank you very much for participating in our effort to produce a diagnostic instrument for upper middle and high school mathematics professional development projects. Our aim is to identify meanings and ways of thinking that provide important supports for helping to learn mathematics with understanding.

Our focus on personal mathematical meanings has led us to ask questions that might seem unusual to you. We are interested in your ways of thinking. We have little interest in whether you can provide correct answers. For many items there is no correct answer. As such, please write in a way that conveys your thinking.

Your packet contains 44 items. You will have 2 hours to respond. Write in ink (pencil does not scan well). If you write something that you wish to change, please cross it out with a single stroke. Do not black out what you write—even your initial thoughts can give us important information about how an item might be improved.

The session will begin with an 18-minute section in which we will present visualizations for you to consider as you answer questions about them.

It is important that you attempt each item. If you do not understand a question, or you do not know how to answer it, then please write a *brief* statement explaining your confusion.

Again, thank you very much for your participation.

Patrick W. Thompson

Professor, Mathematics Education

Principal Investigator, Project Aspire

[1] Supported by NSF Grant No. MSP-1050595.

Advice Regarding Your Answers

All functions are defined over the real numbers unless stated otherwise.

You have limited time to answer these questions, and we want you to answer as many as you can. It is therefore important that you use your time wisely. Here are some suggestions:

- If you have a choice between doing messy calculations and writing uncalculated expressions, write uncalculated expressions.
  - Write $\sqrt{1.35^8 - 2.75^2}$ instead of 1.86277.
  - Write 3^{12} instead of 531,441.
  - Write $(x-4)(2x+3)$ instead of expanding it into $2x^2 - 5x - 12$.
- If you find yourself writing a long essay to explain "why", stop. Regroup. Then say a few more words. After a point, length of explanations is negatively correlated with clarity of exposition.

In short, use your time wisely.

Pause Here and Wait for Further Instructions

I.R.17.v4.A.Racing Cars

Part 1: State the meaning you would like your students to have.

Part 2: Is your stated meaning related to the animation? Explain.

I.P.08.v8.A.Proportional Figures

Part 1
Express the value of p in terms of the value of j and the measures of other parts. Put your answer in the box.

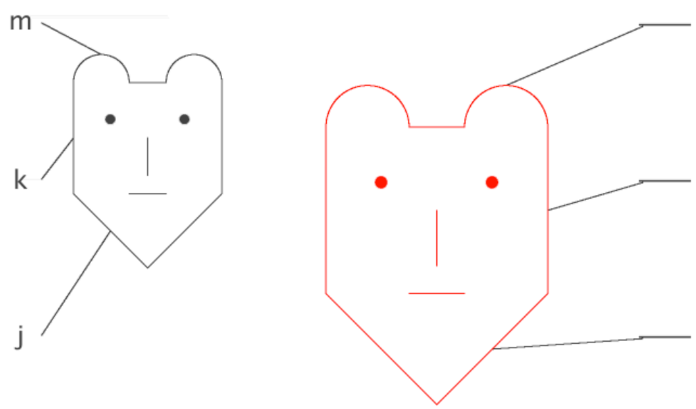
$p =$ [ ]

Part 2
Express the value of B in terms of the value of A and the measures of other parts. Put your answer in the box.

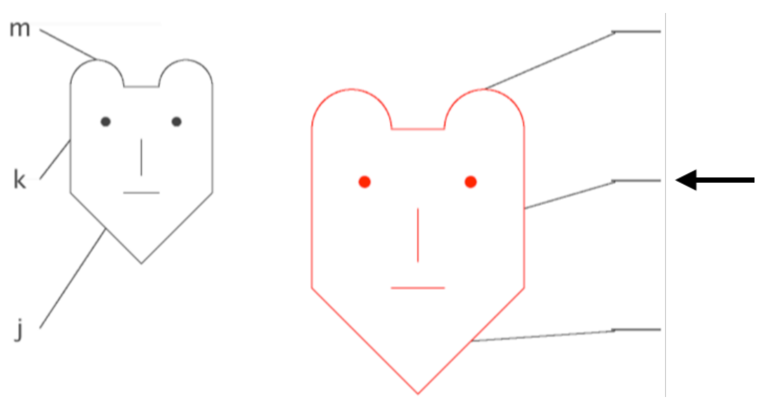
$B =$ [ ]

I.FN.15.v5.A.Represent varying length

Part 3
Represent each indicated part's length in the right figure to reflect that it varies with the number of seconds since the animation began.

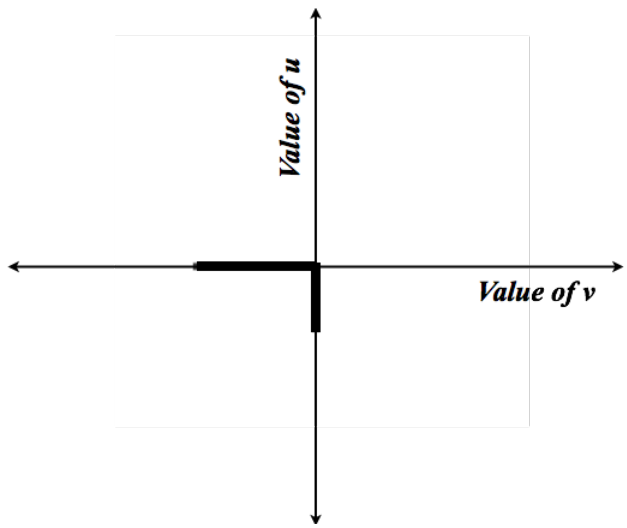


Express the length of the right figure's cheek in terms of the lengths of other parts.



I.C.12.v1.A.Graph Covarying Magnitudes

The values of *u* and *v* vary. The animation shows the same variation repeatedly. Sketch a graph of the value of *u* relative to the value of *v* in the diagram below. The diagram presents the initial values of *u* and *v*.



I.P.09.v6.A.Varying Proportion

Part 1

Complete the definition of the side length $o(t)$ so that it is a function of the number of seconds since the animation began.

$o(t) =$

I.FN.20.v1T
Part 2

Constant of proportionality at 3.25 seconds =

I.R.23.v3.A.Falling Balls

Circle the letter for the phrase that should go in the blank.

The average speed of Ball 1's fall _______ the average speed of Ball 2's fall.

(a) is less than

(b) is equal to

(c) is greater than

(d) cannot be compared to

(e) I don't know.

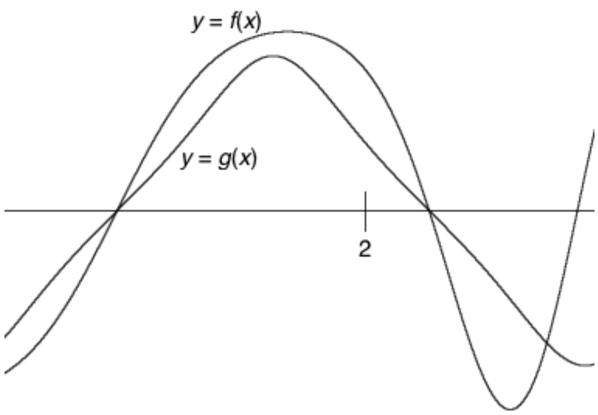
Please explain your selection.

You have completed the animations section of this instrument. The following pages contain written questions for your consideration.

You may turn this page and start answering questions!

I.M.03.v2.T.Plot value of f(2)+g(2)

Show the value of $f(2) + g(2)$ at an appropriate place below.



I.R.14.v4.T.Difference from Rate MC

Consider a non-linear function defined on the interval 7.3 to 7.6. The function's average rate of change over that interval is 4. What is the difference between the value of the function at $x = 7.6$ and the value of the function at $x = 7.3$?

Select the best answer.

a. 0.3×4
b. 4
c. $0.3\ /\ 4$
d. $4\ /\ 0.3$
e. $7.6 - 7.3$
f. Not enough information.
g. I don't know.

I.S.01.v8.T.Associative Property

Δ is a closed operation on the real numbers with the following property.

For all real numbers a, b, and c, $(a \Delta b) \Delta c = a \Delta (b \Delta c)$.

Let u, v, w, and z be real numbers. Can this property of Δ be applied to the expression below? If yes, demonstrate as if to students. If no, explain to students why it cannot.

$$(u \Delta v) \Delta (w \Delta z)$$

I.R.08.v6.T.San Diego to El Centro

A car went from San Diego to El Centro, a distance of 90 miles, at 40 miles per hour. At what speed would it need to return to San Diego if it were to have an average speed of 60 miles per hour over the round trip?

I.R.08.v6.T.San Diego to El Centro, page 2

Part B. A round trip of 180 miles at an average speed of 60 mi/hr will take 3 hours. Is your answer on the prior page consistent with this fact?

☐ Yes ☐ No ☐ I Don't Know

Part C. If you checked "No" or "I Don't Know", then please re-work the problem here. ***Do not*** cross out your work on the prior page.

I.S.14.v5.T.Put parentheses in expression

Add ***all*** parentheses to the expression below to show its structure according to the order of operations.

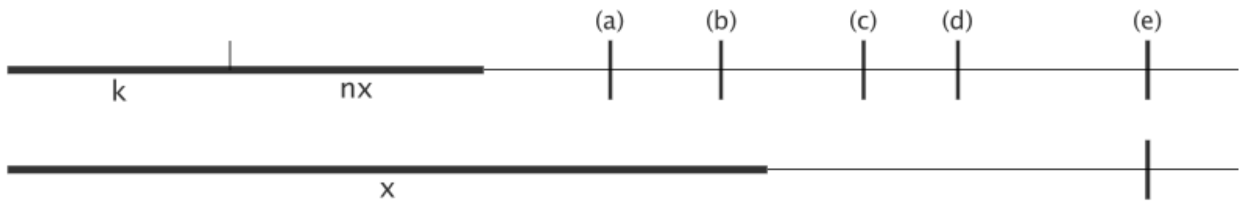
$$x - 5 / y - z + 4 * q + 5 / u * h$$

I.P.02.v7.T.Proportionality in Teaching MC

In the diagram below, the lower bar represents a value of x. The upper bar represents a value of $nx + k$ for some numbers n and k.

Part A. The lower bar will be stretched to end at the hash mark on its line. Circle the hash mark at which you think the upper bar will end.

If you decide that none of the upper hash marks is correct, then place your own. If you do not know where to place a hash mark, select (f) I don't know.



(f) I don't know.

Part B. Explain how you decided upon the correct mark for the upper bar.

I.FN.07.v5.T.Understanding input variables

Here are two function definitions.

$w(t) = \sin(t-1)$ if $t \geq 1$

$q(s) = \sqrt{s^2 - s^3}$ if $0 \leq s < 1$

Here is a third function c, defined in two parts, whose definition refers to w and q. Place the correct letter in each blank so that the function c is properly defined.

$$c(v) = \begin{cases} q(\_\_\_) \text{ if } 0 \leq \_\_\_ < 1 \\ \\ w(\_\_\_) \text{ if } \_\_\_ \geq 1 \end{cases}$$

I.FN.07.v5.T.Understanding input variables, page 2

Part B

James, a student in an Algebra 2 class, defined a function *f* to model a situation involving the number of possible unique handshakes in a group of *n* people. He defined *f* as:

$$f(x) = \frac{n(n-1)}{2}$$

According to James' definition, what is $f(10)$?

I.R.20.v3.T.Relative rates MC

Every second, Julie travels j meters on her bike and Stewart travels s meters by walking, where $j > s$. In *any* given amount of time, how will the distance covered by Julie compare with the distance covered by Stewart?

a. Julie will travel $j - s$ meters more than Stewart.
b. Julie will travel $j \cdot s$ meters more than Stewart.
c. Julie will travel j / s meters more than Stewart.
d. Julie will travel $j \cdot s$ times as many meters as Stewart.
e. Julie will travel j / s times as many meters as Stewart.
f. I don't know.

I.FN.16.v2.T.Functions reference functions

The functions *f*, *g*, and *h* are defined below.

$$f(u) = u^2 - 1$$
$$g(s) = 1 + \frac{f(2s+1)}{2}$$
$$h(r) = g(r/3) - 1$$

What is $h(9)$? Show your work.

I.S.11.v3.T.Quirky Quadratic

Mrs. Rosenthal was teaching a unit on solving exponential equations. The book contained this equation:

$$e^{2x} - 5e^{x} + 6 = 0$$

How must students think of this equation in order to start solving it?

I.V.15.v2.T.Thinking about variables and equations

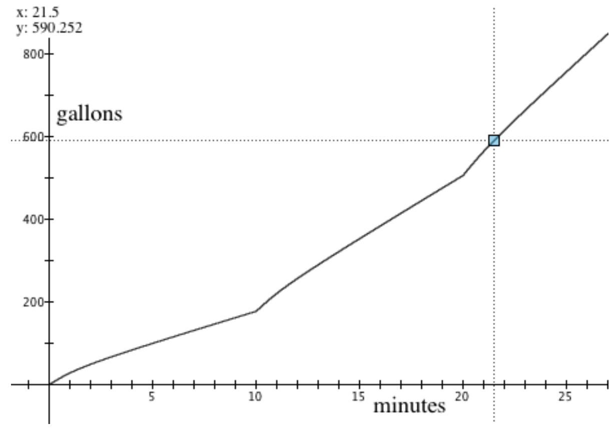
In a lesson on linear equations, Darren, a student in Mrs. Bryant's class, asked, "Mrs. Bryant, why do they call x a variable in $3x + 7 = 12$ when x can be only one number? Didn't you say that a letter that stands for just one number is a constant?"

How would you respond to Darren's dilemma?

I.FM.11.v3.T.Multiple pumps fill a pool

Several machines pump water into a pool. The machines operate independently of each other and get less efficient over time. The number of gallons pumped by any machine after t minutes of operating is given by $w(t)$, where

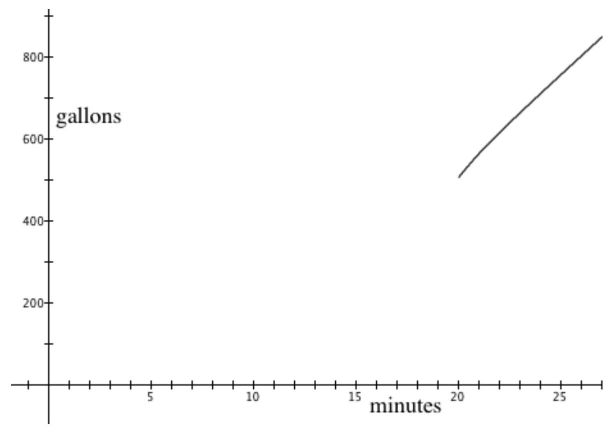
$$w(t)=\begin{cases}\left(30-15e^{-2/t}\right)t \text{ if } t>0\\ 0 \text{ if } t\le 0\end{cases}$$



The graph of $y=w(x)+w(x-10)+w(x-20)$, $x\ge 0$, is given to the right. It shows the number of gallons in an initially empty pool that was filled with three pumps starting 10 minutes apart.

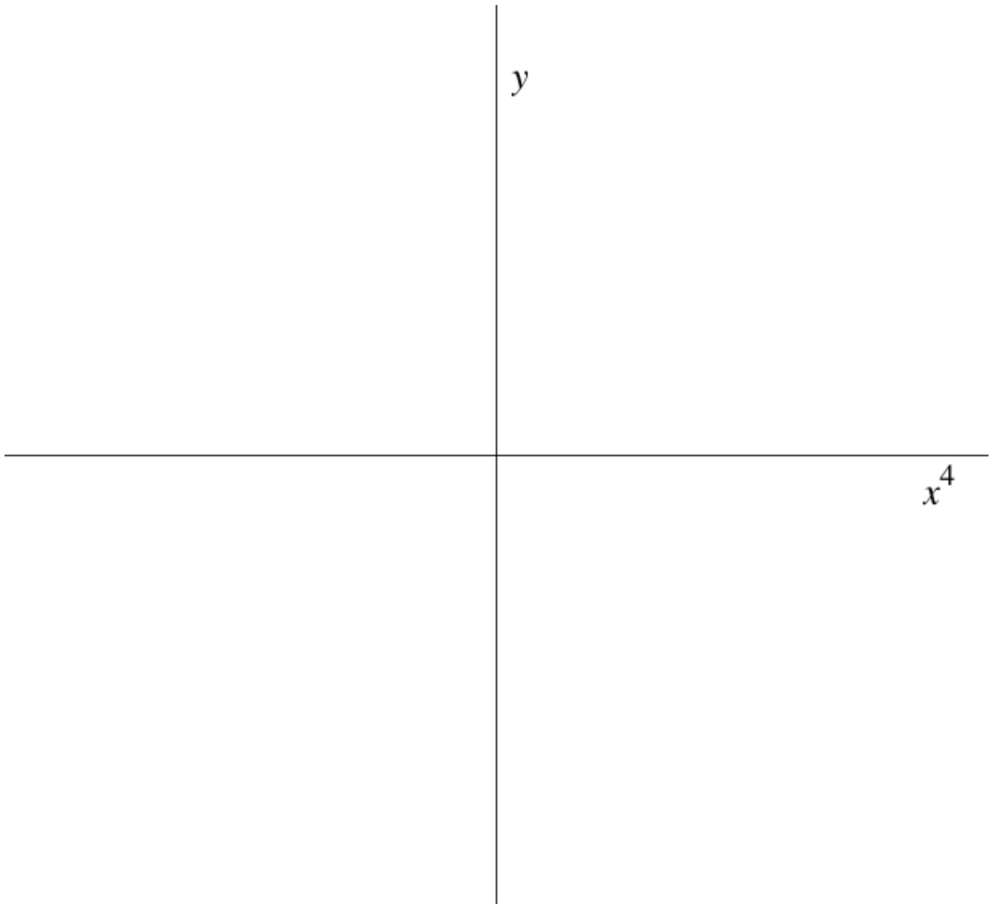
A. The point (21.5, 590.252) is highlighted on the graph. What does this point represent about the situation?

B. Billy defined w as $w(t)=\left(30-15e^{-2/t}\right)t$ if $t>0$, omitting "0 if $t\le 0$". The graph of $y=w(x)+w(x-10)+w(x-20)$, $x>0$, using Billy's definition of w, appears to the right. Why are pieces missing?



I.S.07.v4.T.Thinking in arguments—x^4 on horiz axis

Suppose values of x^4 are on the horizontal axis of a coordinate system and y is on the vertical axis. Sketch the graph of $y = x^2 + 5$.



I.F.10.v1.T.Where g is defined

Suppose that f is defined on the interval $x = 0$ to $x = 5$.

Consider the function g, where $g(t) = f(t - 5)$.

Over what interval is g defined?

I.FR.01.v6.T.Willie chases Robin

Robin Banks ran out of a bank and jumped into his car, speeding away at a constant speed of 50 mi/hr. He passed a café in which officer Willie Katchim was eating a donut. Willie got an alert that Robin had robbed the bank, jumped into his patrol car, and chased Robin at a constant speed of 65 mi/hr. Willie started 10 minutes after Robin passed the café.

Part A. Let *u* represent the number of hours since Robin passed the café. Write an expression that represents the number of hours since Willie left the café.

Part B. Here are two functions. They each represent distances between Willie and Robin.

$$f(x) = 65x - 50\left(x + \tfrac{1}{6}\right), x \geq 0.$$
$$g(x) = 65\left(x - \tfrac{1}{6}\right) - 50x, x \geq 1/6.$$

i) What does x represent in the definition of f?

ii) What does x represent in the definition of g?

Part C. Functions f and g both give a distance between Willie and Robin after x hours. But $f(1) = 6.67$ and $g(1) = 4.17$. Why are $f(1)$ and $g(1)$ not the same number?

I.S.17.v1.T Structure in Frame of Reference and I.FN.19.v1T

Part D. Fill in the blank to make this statement true: $f(w) = g(\_\_\_\_\_)$

I.S.05.v1.T.Solution from identical structure

The equation $3x^5 - 2x^2 + 4 = 0$ has $x = -0.942$ as a solution.

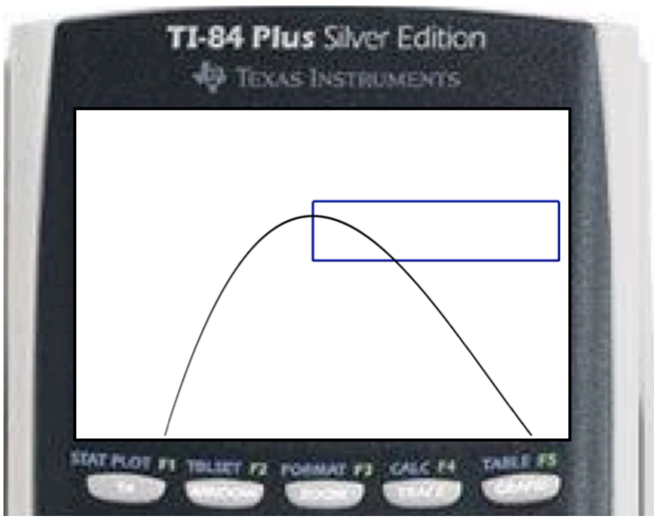
What is a solution to $3(x+1)^5 - 2(x+1)^2 + 4 = 0$?

I.P.05.v6.T.Zooming a graph

Billy is using his calculator to zoom in on part of a graph by drawing a rectangle as shown at the left, below. His calculator will fill the screen with the part of the window contained by the rectangle. Use the screen area of the calculator on the right to sketch the zoomed graph that will appear on Billy's screen.

Please use the right-hand calculator's screen area to show what will appear on it.

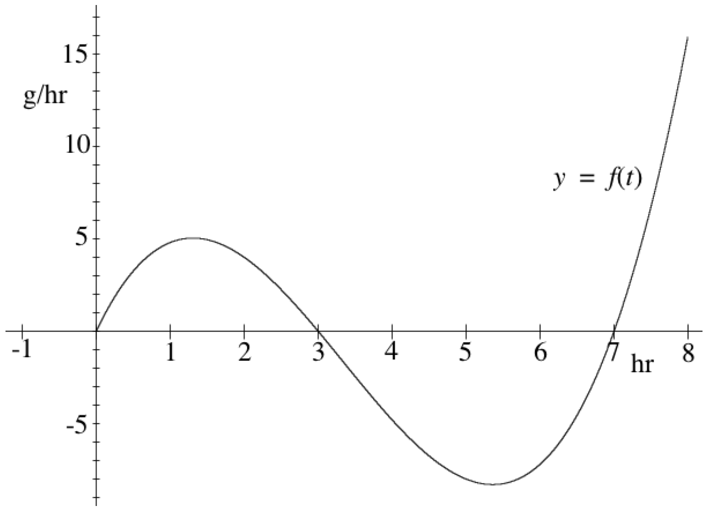
Before Zooming



After Zooming

I.R.07.v6.T.Inc or Dec from rate MCO

The values of function *f* give the rate of change (in grams/hr) of a bacterial culture's mass *t* hours after measurements began.



Over what intervals within the first 8 hours is the culture's mass increasing? Explain.

a) $0 < t \leq 1.4$ and $5.5 < t \leq 8$

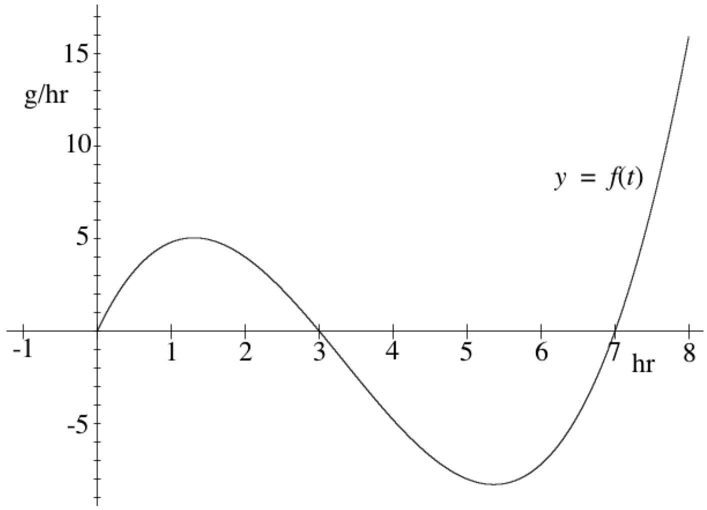
b) $0 < t < 8$

c) $0 < t < 3$ and $7 < t \leq 8$

d) None of the above. My answer is ______________________________

e) I don't know

Part B. The graph from the prior page is repeated below. Highlight the point (2.5, 2.25) on the graph of *f*. What does this point represent?



Part C.

Would you like to change your answer to the question on the prior page? Make the appropriate selection.

a) $0 < t \leq 1.4$ and $5.5 < t \leq 8$

b) $0 < t < 8$

c) $0 < t < 3$ and $7 < t \leq 8$

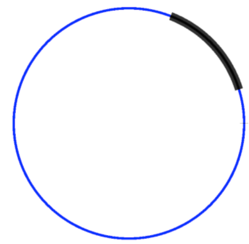
d) None of the above. My answer is ______________________________

e) I don't know

f) I do not want to change my answer.

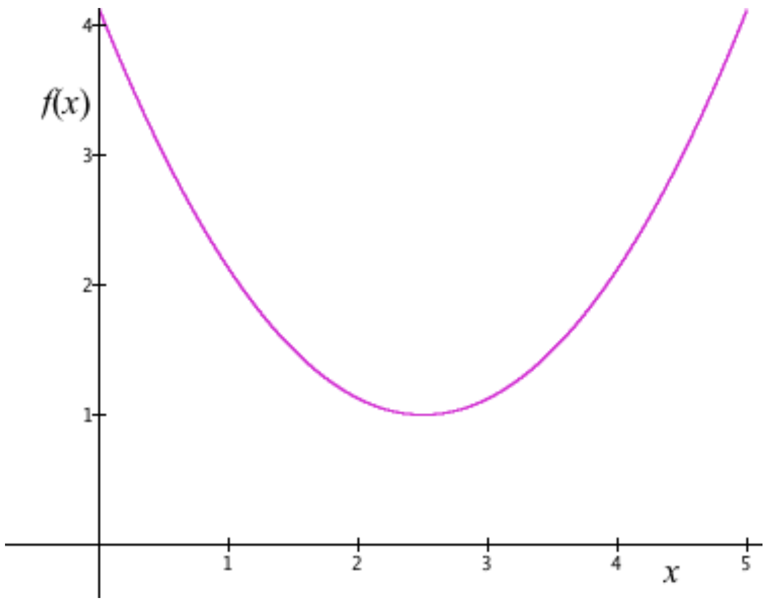
I.M.01.v1.T.Change unit of arc measure

In Nerdland they measure lengths in Nerds. The highlighted arc measured in Nerds is 12 Nerds. In Rapland they measure lengths in Raps. One Rap is 3/4 the length of one Nerd. What is the measure of the highlighted arc in Raps?



I.C.03.v6.T.Covariation and change MC

The graph below is of a function *f* over the interval [0,5].



For small equal increases of the value of *x* starting at $x = 1$ and ending at $x = 2$, the corresponding changes in the value of *f* are….

a) positive and increasing

b) positive and decreasing

c) negative and increasing

d) negative and decreasing

e) I don't know

I.C.03.v5.T.Covariation and change MC, page 2

Part B. Is this sequence increasing or decreasing? -10, -9.5, -9, -8.5, … ___________________

Part C. Would you like to change your answer to the question on the prior page? Circle the appropriate selection.

a) positive and increasing

b) positive and decreasing

c) negative and increasing

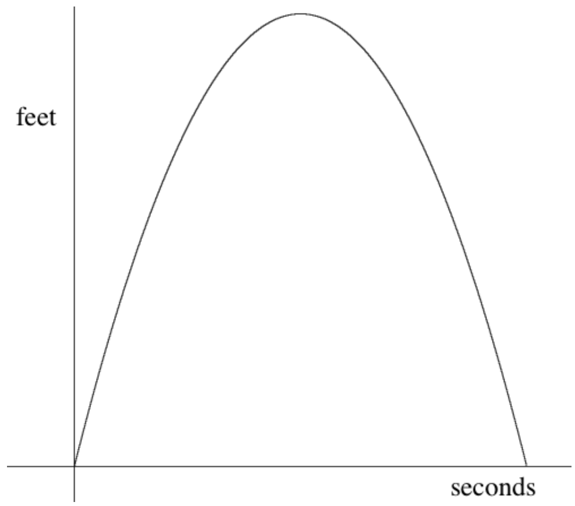
d) negative and decreasing

e) I don't know

f) I do not want to change my answer

I.FM.14.v6.T.Ball tossed vertically MC

A ball was tossed straight up and was caught at the same height it was released. The height of the ball above the point it was released is modeled by $h(t) = 40t - 16t^2$, where t is the number of seconds since it was tossed.



Select ***all*** the interpretations of the graph of h that you would accept as a student's ***main*** meaning.

a. The graph shows the path of the ball as it traveled up and then down after being tossed.
b. The graph shows the ball's height above its release point and how long it was in the air.
c. The graph shows the ball's height at each moment in time during its flight.
d. The graph shows that the ball moved to the right as it went up and down.
e. I don't know.

I.FR.02.v2.T.Nicole chases Ivonne

Ivonne and Nicole jog together at the local track because they run at the same speed. Ivonne arrived early and started running before Nicole arrived and started running. Ivonne had run A laps when Nicole had run B laps.

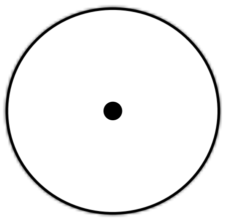
Later, Ivonne thinks, "When I have run C laps, Nicole will have run __________ laps."

Later yet, Nicole thinks, "When I have run D laps, Ivonne will have run __________ laps."

Fill in each blank with an appropriate expression.

I.FM.13.v2.T.Splash in a puddle

Hari dropped a rock into a pond creating a circular ripple that spread outward. The ripple's radius increases at a non-constant speed with the number of seconds since Hari dropped the rock. Use function notation to express the area inside the ripple as a function of elapsed time.



I.P.04.v5.T.Meaning of Slope

Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04.

Convey to Mrs. Samber's students what 3.04 means.

I.P.04.v5.T.Meaning of Slope, page 2

Part B.

Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04.

A student explained the meaning of 3.04 by saying, "It means that every time x changes by 1, y changes by 3.04." Mrs. Samber asked, "What would 3.04 mean if x changes by something other than 1?"

What would be a good answer to Mrs. Samber's question?

I.R.21.v4.T.Olympics Ave Speed

Mrs. O'Neill gave this problem to her students.

> Inger Miller ran the 200-meter dash in 1999 with a time of 21.77 seconds. Alice Cast ran the 200-meter dash in 1922 with a time of 27.80 seconds. Suppose that they ran their races against each other. Approximately how many meters would have been between them when Inger Miller crossed the finish line?

Tanya explained her reasoning this way:

> Assume they ran at constant speeds. 21.77 is about 78% of 27.80. So Alice would have been about 44 meters behind Inger.

Explain to the class what 21.77 being 78% of 27.80 has to do with solving this problem.

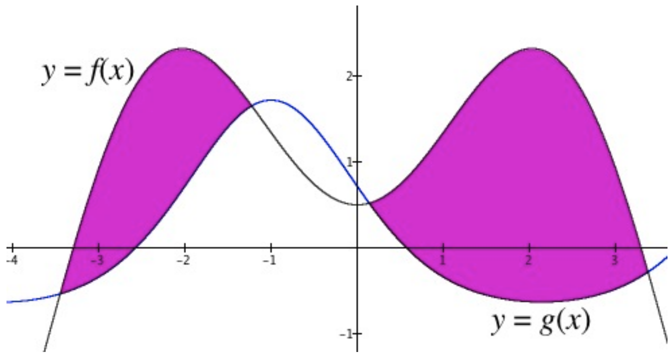
I.S.16.v1.T.Find missing function

f, *g*, and *h* are functions such that $h(x) = f\left(g(x)\right)$. If $f(x) = x^3 + 1$ and $h(x) = 2x - 1$, what is $g(x)$?

I.F.01.v8.T.Describe region MC

Select the choice below that fills in the blank so that the resulting statement describes the points that make up the highlighted region (including its boundary).

The set of all ordered pairs of real numbers (x, y) such that ________________________.



a. $g(x) \le f(x)$

b. $g(x) \le y \le f(x)$

c. $g(x) \le (x,y) \le f(x)$

d. $f(x) - g(x)$ when $g(x) \le f(x)$

e. $-3.3 \le x \le -1.25$ and $-0.5 \le y \le 2.25$ AND $0.15 \le x \le 3.25$ and $-0.75 \le y \le 2.25$

f. I don't know.

I.FN.06.v4.T.Function definitions

A function *f* over the real numbers is defined below. Circle what you would like your students to think represents the function's output.

$$f(x) = x(11 - 2x)(8.5 - 2x)$$

Here is a second function. Circle the function definition.

$$g(x) = x \sin e^{x}$$

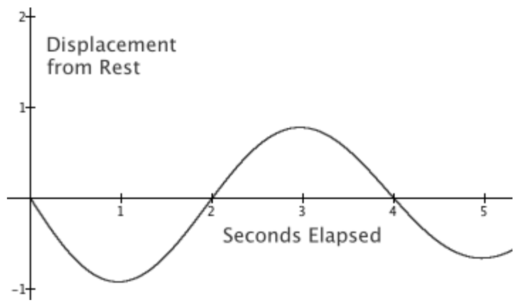
I.M.02.v1.T.Liters to gallons

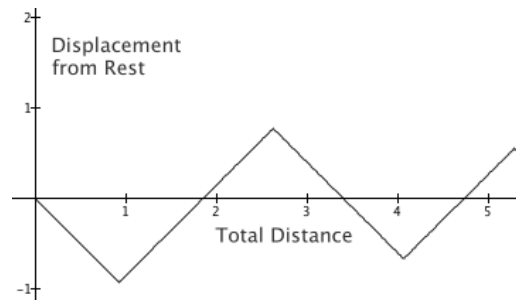
A container has a volume of *m* liters. One gallon is $\frac{189}{50}$ times as large as one liter. What is the container's volume in gallons? Explain.

I.C.01.v12.T.Bouncy Ball MCO

A ball is hanging by a 10-foot rubber cord from a board that is 20 feet above the ground. The ball is given a sharp push downward and is left free to bob up and down.

The graph on the left represents the ball's *displacement from its resting point* in relation to its *time elapsed* since being pushed. The graph on the right is the ball's *displacement from its resting point* in relation to its *total distance traveled* since being pushed.





Part A. Why is the graph of *Displacement from Rest* versus *Total Distance* made of straight segments? (*Select the **best** answer.*)

a. Distance, which is a linear measurement, must be represented with a straight segment.
b. A change of one in displacement corresponds to a change of one in total distance.
c. The ball's displacement in either direction is correlated with changes in total distance.
d. Any small change in displacement in a direction is the same magnitude as the change in total distance.
e. The graph represents the motion of the ball. The graph is made of line segments, demonstrating that the ball travels in a linear fashion.
f. I don't know.

Part B. Select ***all*** the statements about the graph on the right that are true.

a. The graph's cusps show that the ball changes direction suddenly.
b. The graph's straight segments show that the ball moves at constant speeds for periods of time.
c. The graph's cusps show that the ball changes speeds suddenly.
d. None of a, b, or c is true.
e. I don't know.

I.F.08.v7.T.Common domain

Here are four lines of text typed into a graphing program.

$u(p) = \dfrac{3}{p-1}$ if $3 < p < 12$
$w(r) = 2r + 10$ if $0 < r < 7.5$
} These two lines of text define the functions u and w and restrict their inputs

$A(v) = u(v) \cdot w(v)$ } This line of text defines the function A as a product of u and w

$y = A(x),\ 5 < x < 12$ } This line of text tells the program to graph A over the interval $5 < x < 12$

Over what interval on the x-axis will a graph appear?

I.S.15.v3.T.Non-equivalence-preserving transformation

Mr. Abri posed this problem to his students.

Solve for x in $xn - 2x = xn + x$.

Baruti submitted this work:

$$xn - 2x = xn + x$$
$$x(n-2) = x(n+1)$$
$$\frac{x(n-2)}{x} = \frac{x(n+1)}{x}$$
$$n - 2 = n + 1$$
$$-2 = 1 \text{ Contradiction!! No solution.}$$

Explain Baruti's mathematical error, if there is one.

I.FN.18.v5.T.Function value from input vs from argument MCO

Given: $f(x+1) = 3(x+1)^2 + 4(x+2) + 3$

What is $f(2)$?

Select your answer from this list.

a) 46

b) 23

c) 31

d) 27

e) 71

f) None of the above. My answer is ___________________________

g) I don't know.

I.R.22.v3.T.What does 'over' mean?

A college science textbook contains this statement about a function *f* that gives a bacterial culture's mass at moments in time.

The change in the culture's mass over the time period Δx is 4 grams.

Part A. What does the word "over" mean in this statement?

Part B. Express the textbook's statement symbolically.

I.FN.03.v10.T.b or b-5 MC

h is a strictly increasing **function** defined for all real numbers. $h(b - 5) = 9$ for some number *b*.

Which of $(b, 9)$ or $(b - 5, 9)$ is on the graph of $y = h(x - 5)$ when graphed on a calculator? Explain.

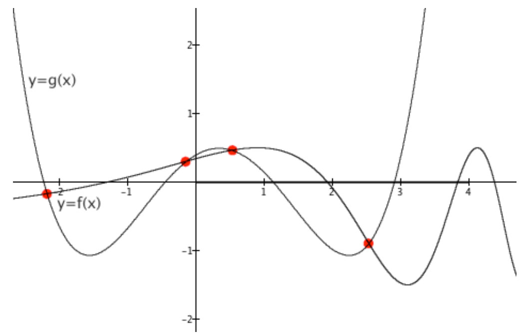
Select the best answer and explanation.

a. $(b, 9)$ is on the graph because *b* is the input that gives 9 as the output .
b. $(b, 9)$ is on the graph because $b - 5$ is the input to *h* and the function is strictly increasing.
c. $(b, 9)$ is on the graph because when we solve for *h*, $h = \frac{9}{b-5}$, making $y = \left(\frac{9}{b-5}\right)(x-5)$, for which $x = b$ produces a value of 9.
d. $(b - 5, 9)$ is on the graph because $b - 5$ is in the *x* position and 9 is in the *y* position.
e. $(b - 5, 9)$ is on the graph because $b - 5$ is the input that gives 9 as the output.
f. I don't know.

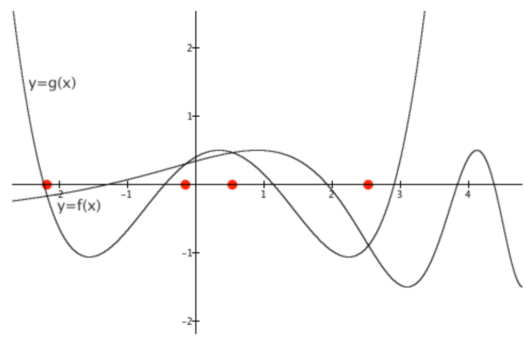
I.F.12.v2.T.Solutions to f(x)=g(x)

Three students drew graphs to represent the solutions to an equation of the form $f(x) = g(x)$. Which of these three ways of thinking would you prefer your students to have about solutions to $f(x) = g(x)$? Why?

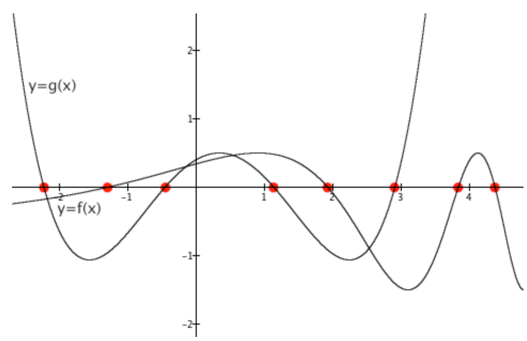
Student 1:
Since the solutions to $f(x) = g(x)$ are ordered pairs we should highlight the points of intersection on the graphs of f and g.



Student 2:
Since the solutions to $f(x) = g(x)$ are values of x, we need to highlight the x values of the points of intersections on the graphs of f and g.



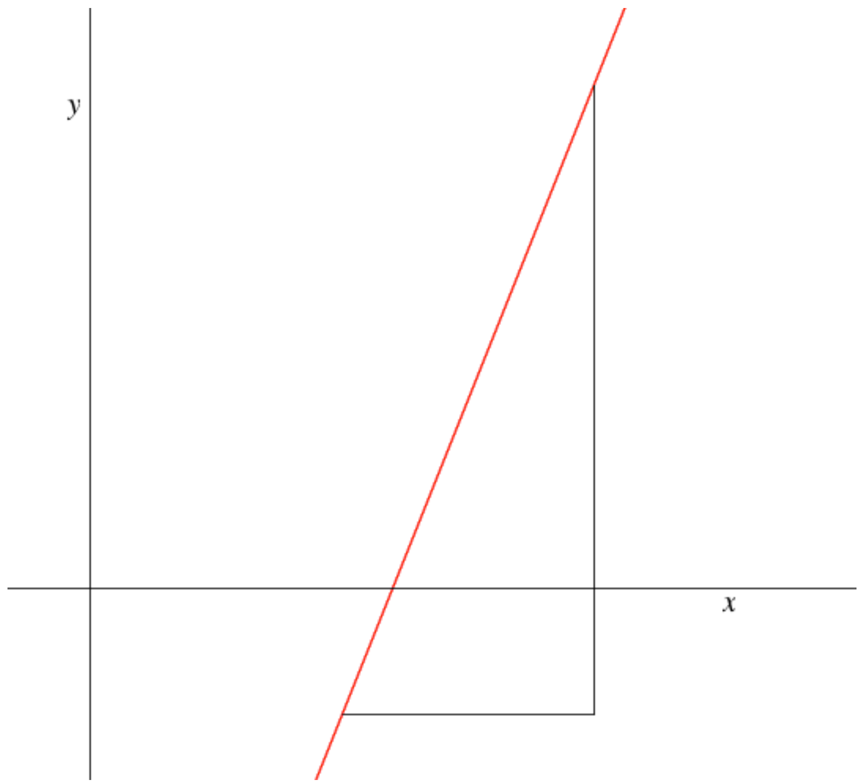
Student 3:
A solution is where a graph crosses the x-axis, so we need to highlight those places where either graph crosses the x-axis.



Your answer and explanation:

I.M.04.v5.T.Slope from blank graph

Part A. There are two quantities *P* and *Q* whose values vary. The measure of *P* is *y* and the measure of *Q* is *x*. *y* and *x* are related so that $y = mx + b$. The graph of their relationship is given below, with *x* and *y* in the same scale. What is the numerical value of *m*?



Part B. What would be the numerical value of *m* if the *y*-axis were stretched so that the distance between 0 and 1 is 2 times as large as the original?

I.S.10.v3.T.Continued Fraction

Let C be defined as

$$C = 1 + \cfrac{3}{1 + \cfrac{3}{1 + \cfrac{3}{1 + \cfrac{3}{1 + \cdots}}}}$$

Then C can be rewritten as (write something below the horizontal line):

$$C = 1 + \cfrac{3}{1 + \cfrac{3}{\rule{5em}{0.4pt}}}$$

Project Aspire: Defining and Assessing Mathematical Meanings for Teaching secondary mathematics

Demographic Data

Please complete this form if you have not already filled one out.

Your gender: Female Male Other/Decline

School Location: Urban Suburban Rural

Name of your mathematics teaching credential: ____

Degrees

Undergraduate degree: (e.g., BA, BS, BAE) ____ Year: _____

Undergraduate Major: (e.g., Math, Math Ed, Psychology) _____________________________

Highest Graduate degree: (e.g., MS, MA, MAT, EdD, PhD) ___________________________ Year: _____

Highest Graduate Major: (e.g., Math, Math Ed, Leadership) ___________________________

Mathematics Teaching Experience:

If you taught the same course to n different classes in the same year, count the course as having taught it n times.

Complete Years taught: ______

Algebra I course: _____ times

Algebra II course: _____ times

Geometry course: _____ times

Precalculus course: _____ times

Calculus AB course: _____ times

Calculus BC course: _____ times

Differential Equations course: _____ times

College Algebra course: _____ times

Pre-algebra course: _____ times

Other: _____________________________________ ___ times