Rounding numbers: Ptolemy's calculation of the Earth-Sun distance

Author(s): Christián C. Carman

Source: *Archive for History of Exact Sciences*, March 2009, Vol. 63, No. 2 (March 2009), pp. 205-242

Published by: Springer

Stable URL: http://www.jstor.com/stable/41134305

Arch. Hist. Exact Sci. (2009) 63:205–242
DOI 10.1007/s00407-008-0038-6

Rounding numbers: Ptolemy's calculation of the Earth–Sun distance

Christián C. Carman

Received: 10 October 2008 / Published online: 10 December 2008

Abstract In this article, I analyze the coincidence of the prediction of the Earth–Sun distance carried out by Ptolemy in his *Almagest* and the one he carried out, with another method, in the *Planetary Hypotheses*. In both cases, the values obtained for the Earth–Sun distance are very similar, so that the great majority of historians have suspected that Ptolemy altered or at least selected the data in order to obtain this agreement. In this article, I will provide a reconstruction of some way in which Ptolemy could have altered or selected the data and subsequently will try to argue in favor of its historical plausibility.

1 Introduction

In the *Almagest*[1] Ptolemy obtains the lunar parallax in a highly theoretical way. He calculates the angular position of the Moon at a certain moment based on his model and compares it with an observed position. Given that the calculations of his model should provide him with the results seen from the center of the Earth, the difference between the two values is precisely the lunar parallax. Based on this parallax, Ptolemy

[1] I will use Toomer [1984] 1998. The Roman number indicates the book, Arabic numbers indicate the chapter. The page number in Heiberg's edition is given after the first semicolon, and that in Toomer's translation, after the second semicolon. If the position in Heiberg is not specified, it refers to the first volume. So, H232 refers to page 232 in the first volume, whereas H(2) 232 refers to the same page in the second volume.

Communicated by A. Jones.

C. C. Carman (✉)
CONICET-UNQ, Buenos Aires, Argentina
e-mail: ccarman@unq.edu.ar

is able to calculate that the Moon, at its maximum distance from the Earth (D_L), is at 64;10 terrestrial radii (64^{tr}).[2] Then, using the data he obtains from two lunar eclipses, he finds for the Earth–Sun distance (D_S) a value of 1210^{tr}.

In another work, the *Planetary Hypotheses*,[3] Ptolemy tries to establish the absolute distances of the planets. In the *Almagest*, making use of his deferent and epicyclical system, he was able to establish the proportion between the radius of the deferent (R) and that of the epicycle (r), and if he also considered the eccentric (e), he could calculate the proportion between a planet's maximum and minimum distances. Furthermore, assuming that vacuum does not exist and that there are not any useless things in nature, the maximum distance of a planet (apogee) corresponds to the minimum distance of the immediately superior planet (perigee). If he had one absolute distance and the order of the distances of the heavenly bodies,[4] then taking these proportions into account he would be able to calculate the maximum, mean and minimum distances of each planet. Using the data obtained in the *Almagest*, Ptolemy rounds D_L to 64^{tr}. He therefore puts Mercury's minimum distance at 64^{tr} and, taking Mercury's proportion to be ${}^{88}/_{34}$, he calculates Mercury's maximum distance to be 166^{tr}, which then coincides with the minimum distance of the following planet, Venus. The proportion between Venus' distances is ${}^{104}/_{16}$, so Venus' maximum distance would be 1079^{tr}. But the Sun's mean distance is 1210^{tr}, and the difference between the mean and the minimum distance is ${}^{1}/_{24}{}^{th}$ of the Sun's mean distance, so consequently, the Sun's minimum distance will be 1160^{tr}. There were several reasons to think that the Sun should be located after Venus, and so Venus' maximum distance should be equal to Sun's minimum one and, indeed, the difference is only 81^{tr}, less than 7%. This is a very surprising coincidence. In fact, it is even more surprising if we keep in mind that if Ptolemy had not rounded the values obtained in the *Almagest* and had been more careful with the calculations, Venus' maximum distance would have reached 1189^{tr}, going beyond the Sun's minimum distance by only 29^{tr}, a mere 2.4%. This will be an amazing coincidence, if we consider the difficulties that Ptolemy must have faced in his observations and calculations. It turns even more astonishing if we consider the following detail. In the *Planetary Hypotheses*, Ptolemy slightly corrects Mercury's parameters, and if the calculations are carried out with the corrected parameters, we obtain a value of approximately 1146^{tr} for Venus' maximum distance, with a difference of only 14^{tr}, which represents 1.1%. This is simply incredible.

Therefore, Ptolemy obtains, by means of two extremely theory laden calculations that are however seemingly independent of each other—that is to say, the data of one

[2] When we reproduce values taken from Ptolemy, we will use the colon and semicolon system, popularized by Neugebauer in ([1957] 1969: 13, note 1). For more details, see chapter 1 of Neugebauer [1957] 1969 or Newton 1977: 17–20.

[3] Something extremely curious happened with the *Hypotheses*, which shows that also history as a discipline can product surprising predictions. Cf. Goldstein (1967).

[4] In Almagest IX, 1, Ptolemy says that "since none of the stars has a noticeable parallax (which is the only phenomenon from which the distances can be derived)" there is no other way to know the order of the planets, and he then asserts that "the order assumed by the older [astronomers] appears the more plausible"; that is to say, even if he apparently cannot justify the order here used, he assumes it. For more discussion about that see Sect. 4.2.

calculation do not determine the results of the other—two remarkably similar values for D_S.[5]

Most historians think that Ptolemy somehow selected the data in order to obtain the results what he actually obtained. For example, we can mention van Helden who writes: "The selection of these two eclipses so far in the past, coupled with our knowledge that estimating the magnitude of an eclipse is extremely difficult and that Ptolemy's procedure was sensitive to even small errors in these estimations, indicates that he was probably looking for particular values" (1987: 17). This agreement was suspicious even to Kepler, who asserts: "if anyone seeks very carefully into the method which Ptolemy employed for establishing the distance of the Sun, he will greatly praise the singular ingenuity of the demonstration; but he will pronounce those things which Ptolemy accepted as very suspect, as if provided for the purpose of demonstrating that which Ptolemy had taken from the ancients".[6] Hartner is even harsher with Ptolemy: "We cannot but admire Ptolemy's ingenuity in working out a procedure, at first sight sound, which entails the result he wished to obtain. It is hard to understand, however, that his obvious fake—one of the most remarkable hoaxes in the history of astronomy—was never recognized as such, to the effect that the ratio 19 : 1 remained unquestioned during the ensuing 1500 years or more." (1980: 25). Hartner finishes his article asserting: "There remains the question how Ptolemy actually proceeded to get the result he wished to obtain. The answer is simple". The aim of this article is to offer a possible answer of how Ptolemy could have done this and, as you will see, clearly this is not simple at all.

I will start (part 2) reconstructing in some detail the calculations carried out by Ptolemy both in the *Almagest* as well as in the *Planetary Hypotheses*. In the reconstruction of the *Almagest* calculation (2.1), I will first develop the way in which Ptolemy obtained the Earth–Sun distance value (2.1.1) and then, how he obtained the values involved in the calculation (2.1.2): the Earth–Moon maximum distance (2.1.2.1), and the apparent radius at that distance of both the Moon and the Earth shadow (2.1.2.2). Then I will explain the calculation carried out by Ptolemy in the *Planetary Hypotheses* (2.2). Besides the Moon distance already considered, three sets of data are involved in this calculation: the Sun's eccentric; Venus' deferent, epicycle and eccentric radii; and

[5] The proportion between the Earth–Moon maximum distance and the Earth–Sun mean distance is 18.86 which coincides with the proportion given by Aristarchus of Samos (between 18 and 20). Aristarchus' calculation was carried out in an exact first quarter. In Ptolemy's lunar model, however, the distance of the Moon in quadratures would be almost half the distance in the syzygies. In quadratures, the proportion is 32.93, which goes largely beyond Aristarchus' proportion. So, the coincidence is only apparent. One can object that even if it is true that Aristarchus used a quadrature configuration, in his simple model with no eccentricities, the proportion D_S/D_M is the *same* in any configuration, including syzygy. So Ptolemy might have been interested in comparing his ratio at syzygy with Aristarchus' at syzygy, independent of the values at quadrature. But, the point is that if Ptolemy understood the procedure followed by Aristarchus and the differences between his model and Aristarchus'—and of course he did—he had to realize that Aristarchus' proportion is only valid in the quadratures of his model. Regarding Aristarchus' calculation see Heath [1913] 1997: 329–336; Neugebauer 1975: 634–643; van Helden 1986: 5–11; Dreyer 1953: 182–184. The coincidence is developed in Newton 1977: 199; van Helden 1986: 19; Dreyer 1953: 184–185; Hartner 1980: 24 and 1964: 255; Evans 1998: 73. It is strange that R. Newton, Ptolemy's official enemy, has not noticed that the coincidence is only apparent. Cfr. Newton 1977: 199. Hartner does notice it (1980: 24).

[6] Kepler (1937-): 8, 414, quoted in van Helden (1986): 19.

Mercury's deferent, epicycle and eccentric radii. I will finish part 2 saying something about the independence of the values involved (2.3).

Part 3 will be devoted to the analysis of the accuracy of data and the sensitivity of methods of calculations used by Ptolemy in order to obtain D_S. I will start by analyzing the D_L calculation (3.1) and the hypersensitivity of the *Almagest's* calculation (3.2). In Sects. 3.2.1–3.2.4, I will evaluate the calculation's sensitivity to each value involved in it and in 3.2.5, I will calculate what the values would be if we use other eclipses which were employed in the *Almagest*.

Part 4 is the core of the article. First (4.1), I will offer my conjecture of how Ptolemy actually altered or selected the data. Then, I will present and try to answer some possible objections (4.2). After that, I will offer two supporting facts of my conjecture (4.3): the first related to the calculation of Venus' absolute radius in *Planetary Hypotheses* (4.3.1) and the other linked with the calculation of the Earth's shadow and the Moon's apparent radii at the Earth–Moon minimum distance (4.3.2). I will finish this part reconstructing how Ptolemy could have actually done the alteration or selection of data (4.4).

I will finish the article (part 5) by summarizing the road that Ptolemy might have followed and enumerating the facts my conjecture could be able to explain.

2 The calculations

2.1 The calculation of DS in the *Almagest*

2.1.1 The Earth–Sun distance calculation

I will begin by explaining the calculation that Ptolemy carries out in the *Almagest* in order to obtain D_S. I will do that in the most faithful possible way to Ptolemy's calculations, but I will omit certain details.[7]

It had already been noticed since antiquity that in some solar eclipses, the lunar circumference coincided exactly with that of the Sun. In those eclipses, therefore, the apparent diameters of both the Moon and the Sun are equal. Ptolemy claims that the coincidence takes place when the Moon reaches her maximum distance, not at the mean distance, as previous astronomers—e.g. Hipparchus[8]—had assumed (V,14; H417; 252).

On the other hand, if we consider a lunar eclipse in which the Moon reaches its maximum distance, the lunar apparent diameter will be similar to that of the Sun, and it is also possible to determine the apparent diameter of the circumference of the Earth's shadow at that distance.

[7] A version that is very clear for our own mathematical formation, but for that reason not very faithful, can be found in Evans 1998: 385–389. More accurate explanations can be found in Neugebauer 1975: 101–112, Pedersen 1974: 203–214, van Helden 1986: 15–27, Swerdlow 1968: 41–71. Ptolemy's treatment is in V,11–15; H401-H425; 243–257. In all these citations we also include the calculation of D_L, which, although we will see later on, always appears before in Ptolemy and in his commentators.

[8] Swerdlow proves this in 1969: 291–298.

Aristarchus of Samos has said that the proportion between the diameter of the Earth's shadow and the diameter of the Moon was 2:1. Hipparchus stated that the proportion was 2;30, and Ptolemy obtains, through the study of eclipses, that the proportion is "negligibly less than $2^3/_5$ [2;36]" (V,14; H421; 254). In fact, he affirms that the lunar apparent radius (ρ_{Moon}) is 0;15,40° and that the shadow's apparent radius (ρ_{Sdw}) is 0;40,40°. As we have seen, the diameters of the Moon and of the Sun coincide in a solar eclipse, and therefore the apparent radius of the Sun will also be 0;15,40°.

Based on these data and on D_L, Ptolemy is able to calculate D_S. Let us follow his argument in detail. In Fig. 1, both eclipses are represented together. D indicates the center of the Sun, the Earth is located at N and the Moon at Θ, and the line OPR represents the Earth's shadow at D_L.

Ptolemy reminds us that the angle ENH, which corresponds to the diameter of the Moon, is equivalent to 0;31,20° and therefore

$$\rho_{\text{Moon}} = \Theta\text{NH} = 0;15{,}40°.$$

He also knows that the angle NΘH is a right angle and so

$$D_L = \text{N}\Theta = 64;10^{\text{tr}}.$$

Therefore, ΘH, which measures the real lunar radius (not the apparent), is equal to the tangent of the angle ΘNH multiplied by NΘ. Ptolemy, though, calculates the sine, since with such small angles there is almost no difference. The value given by Ptolemy for ΘH is $0;17{,}33^{\text{tr}}$.

$$\Theta\text{H} = 0;17{,}33^{\text{tr}}.$$

Having calculated the real lunar radius, Ptolemy multiplies it by the ratio of the apparent radii (2;36) and obtains the real radius of the shadow at this distance, that is, PR:

$$\text{PR} = 0;45{,}38^{\text{tr}}.$$

To obtain the value of ΘS, he needs HS, since he already has the value of ΘH and $\Theta\text{S} = \Theta\text{H} + \text{HS}$. In order to obtain HS, Ptolemy carries out the following calculations. On one hand, he knows that

$$\Theta\text{H} + \text{PR} = 1;3{,}11^{\text{tr}}.$$

On the other hand that

$$\text{PR} + \Theta\text{S} = 2^{\text{tr}}.$$

He knows this because PR and ΘS are the extreme sides of a parallelogram, the half height of which is exactly NM, i.e. the Earth's radius, by definition 1^{tr}. NM is in

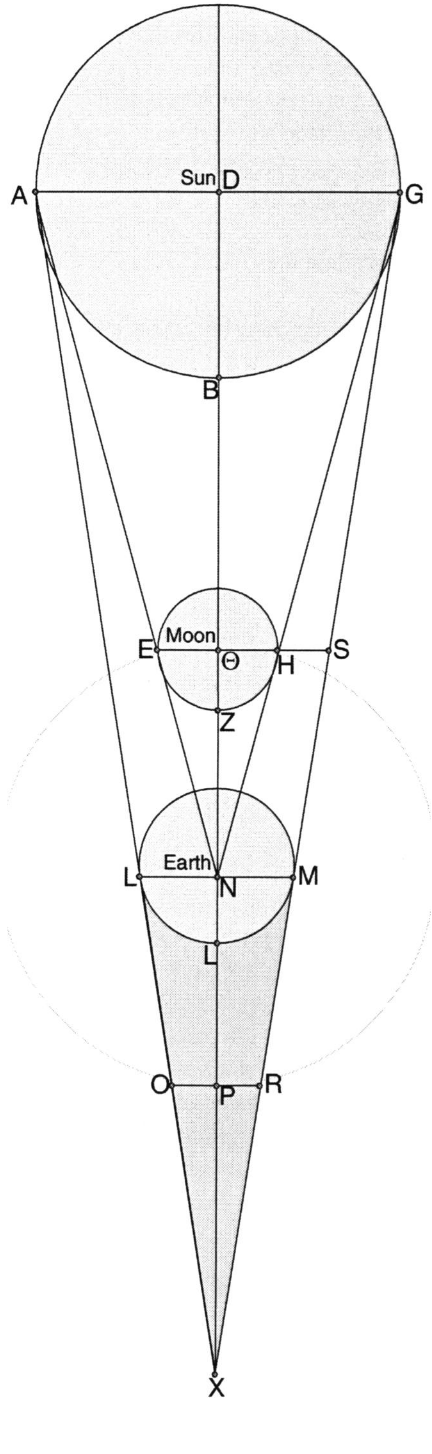


Fig. 1 Eclipse diagram to determine the Sun distance

the middle of the parallelogram because NΘ and NP are equal, since both of them are equivalent to D_L. Now, the sum of the extremes has to equal twice the mean height, and so PR + ΘS = 2. If we subtract (PR + ΘH = 1;3,11^{tr}) from (PR + ΘS = 2^{tr}) we obtain ΘS − ΘH = 0;56,49^{tr}, but ΘS − ΘH is exactly HS, and so

$$\mathrm{HS} = 0;56{,}49^{tr}.$$

Ptolemy knows that ΘD = ND − NΘ, and also that NΘ = 64;10^{tr}. Therefore,

$$\Theta\mathrm{D} = \mathrm{ND} - 64;10^{tr}.$$

Ptolemy also knows that NM/HS = ND/ΘD, and that NM=1^{tr}, and so, after some steps, he obtains that

$$\mathrm{ND} = \frac{64;10^{tr}}{1 - HS}$$

which corresponds to a value of 1209;25,26 that Ptolemy rounds to 1210:

$$\mathrm{ND} = D_S = 1210^{tr}.$$

The distance between the Earth and the Sun, then, is 1210^{tr}. Ptolemy does not specify in the *Almagest*, if 1210^{tr} refers to the maximum, mean or minimum distance, but in the *Hypotheses* he will consider it as the mean distance.[9] As we have seen, in order to obtain it, Ptolemy, apart from some assumptions, has used only three data: (1) the apparent radius of the Sun and the Moon which is equal to 0;15,40°; (2) the apparent radius of the Earth's shadow at this distance, which corresponds to a value of 0;40,40°; and (3) the distance from the Earth to the Moon (64;10^{tr}). The first two values were obtained from the eclipses, which we will comment on later in Sect. 2.1.2.2. We will now analyze how he obtained the distance from the Earth to the Moon.

2.1.2 The calculation of the values involved in the Earth–Sun calculation

2.1.2.1. The Calculation of the Earth–Moon Distance[10] Ptolemy knows that, at its maximum distance, the distance from the Earth to the Moon is equal to the sum of the value of the eccentric plus the radii of both the epicycle and the deferent. Ptolemy has these three values expressed in *parts*, that is to say, in a proportional way.

First, he obtains that, if the radius of the deferent is equal to 60 parts (60^{P}), the radius of the epicycle will be equal to 5;15^{P}. After this, he decides that, in order to explain luni-solar elongations away from syzygy, it is necessary to introduce the eccentric. Thus, that which was previously the radius of the deferent will be broken

[9] Anyway, Ptolemy had said in the *Almagest* that the variation of ρ_{sol} because of D$_S$ is practically imperceptible, for which reason the calculation should be applied to the mean D_S, not to any other.

[10] For bibliographical references see note 7.

up into the new radius of the deferent, which will be equal to $49;41^{\mathrm{P}}$, plus the distance of the eccentric, which will be equal to the remaining $10;19^{\mathrm{P}}$, and so:

$$e_{\mathrm{Moon}} = 10;19^{\mathrm{P}}$$
$$R_{\mathrm{Moon}} = 49;41^{\mathrm{P}}$$
$$r_{\mathrm{Moon}} = 5;15^{\mathrm{P}}$$

The required calculations to obtain these values are complicated and it is not necessary to detail them here. In order to obtain the absolute sizes of the three values, Ptolemy needs to know the proportion between the *parts* and the terrestrial radii. By means of a calculation that I will detail further on, he obtains the proportion $^{59}/_{60}$ [0.9833] which is a rounded number of the more exact value $^{39;45}/_{40;25}$[0.9835]. Therefore, the values which Ptolemy obtains for the real sizes of R_{Moon}, e_{Moon} and r_{Moon}, expressed now in terrestrial radii, are:

$$e_{\mathrm{Moon}} = 10;8^{\mathrm{tr}}$$
$$R_{\mathrm{Moon}} = 48;52^{\mathrm{tr}}$$
$$r_{\mathrm{Moon}} = 5;10^{\mathrm{tr}}$$

The sum of the three values ($e_{\mathrm{Moon}} + R_{\mathrm{Moon}} + r_{\mathrm{Moon}}$) is exactly $64;10^{\mathrm{tr}}$. The only point which still needs to be explained is how Ptolemy obtains the proportion $^{39;45}/_{40;25}$. The numerator ($39;45^{\mathrm{tr}}$) is the value that he obtained for the absolute distance of the Moon, using the calculation of the parallax, on 135 October 1. The denominator ($40;25^{\mathrm{P}}$) is the value that, in parts, the Moon's distance has at that same moment.

Let us see how Ptolemy calculated both values. On 135 October 1, 5 h and 50 min after noon, the Moon was almost at the meridian and Ptolemy measured the angle that separated the Moon from the zenith, obtaining a value of 50;55°. The value calculated by Ptolemy using his lunar theory was 49;48°; and therefore, the parallax was $1;7°(= 50;55° - 49;48°)$.

The calculation by which Ptolemy arrives from the lunar parallax (1;7°) to D_L at that moment ($39;45^{\mathrm{tr}}$) is extremely complicated, but it does not represent a great mystery. In order to calculate a distance, assuming that he knows some angles, Ptolemy needs to build right angles to apply chords tables and Pythagoras' theorem. We saw an example of this when we calculated D_S with the aid of the eclipse diagram. Therefore, the whole problem simply consists in arriving at the distance we need using the angles and distances we have, going from one right-angled triangle to another. The result obtained by Ptolemy is $39;45^{\mathrm{tr}}$.

The calculation of the relative distance follows a style very similar to the previous one, but in this case, the data will be the angles supplied by the Tables (which presuppose the Lunar theory), plus the values of the eccentric and the relative radii of the epicycle and of the deferent. The final result, then, is that at that time on 135 October 1, the Moon was at $40;25^{\mathrm{P}}$ from the Earth.

So at the same moment the Moon was at $40;25^{P}$ and $39;45^{tr}$, the proportion between which Ptolemy rounded to $^{59}/_{60}$.

2.1.2.2. The Calculation of the Moon's apparent radius and the Earth's Shadow It still remains to reconstruct the calculation Ptolemy carried out in order to obtain the value 0;15, 40° for the Moon's apparent radius at its maximum distance (ρ_{Moon}),[11] and 0;40,40° for the apparent radius of the Earth's shadow at that same distance (ρ_{sdw}). The radii could be measured by means of several methods that Ptolemy eschews due to their imprecision. He says that using the *dioptra* described by Hipparchus, he became convinced that the Sun's apparent radius is practically constant and that both the Moon's and the Sun's radii coincide when the Moon reaches its maximum distance (V,14; H417; 252). However, Ptolemy decides to carry out certain extremely theoretical calculations, with the purpose of obtaining more precise values. The idea behind the calculations is very simple. In Fig. 2 the biggest circle represents the circumference of the Earth's shadow; in both parts of the figure, the Moon (the smaller circumference centered in P) is partially immersed in the shadow. The horizontal line passing by the center of the Earth's shadow represents the ecliptic. Evidently, the center of the shadow will be in this line, since the shadow is the effect of the Sun's light, which travels along the ecliptic. The Sun, therefore, will be exactly at 180° from A. The Moon does not follow the path of the ecliptic, it rather moves along its own orbit, which crosses the ecliptic at two points (nodes) at an angle of 5°. Clearly, the eclipses have to take place in proximity to these nodes, because only in those places are the Moon and the shadow close enough in latitude to overlap. The lunar model elaborated by Ptolemy allowed him to calculate, at any moment, the distance between the node and the center of the Moon (line ☊P or ☋P). That line, at the moment of larger occultation, forms a right angle with the center of the shadow (A), which could also be calculated knowing the Sun's exact position. Hence, Ptolemy was able to obtain, with this right-angled triangle, the value of AP. Given that we know how much the Moon was submerged in the shadow (by means of the magnitude of the eclipse), it is easy to obtain the radii we are looking for. In the lower part of Fig. 2, for example, precisely half of the Moon is submerged in the shadow and so the line AP is equal to the shadow's radius. In the upper drawing, we see that only one quarter of the Moon is submerged; therefore, AP is equal to the shadow's radius, which we already know, plus 1/4 of the Moon's diameter or 1/2 of its radius.

Let us now analyze the calculations in detail. Ptolemy uses two eclipses: the first one took place on—620 April 21, (n° 4 of our Chart) and the second one on—420 July 16 (n° 5). The first one is represented in the upper part of Fig. 2, and the second in the lower part.

The value which Ptolemy obtains for ☋P in the first eclipse is 9;20°. Given that we know that the angle at ☋ is 5°, we can obtain AP, which is equal to $\tan(5°) \cdot 9;20°$.

[11] As the apparent radius, contrary to the real one, varies with the distance, we should represent it with $\rho_{\mathrm{mx(Moon)}}$, but as I will often reference this value, I will continue writing it as ρ_{Moon}, assuming that we refer to the apparent radius at the Moon's maximum distance. I will specify when I refer to the radii at the mean or at the minimum distance.

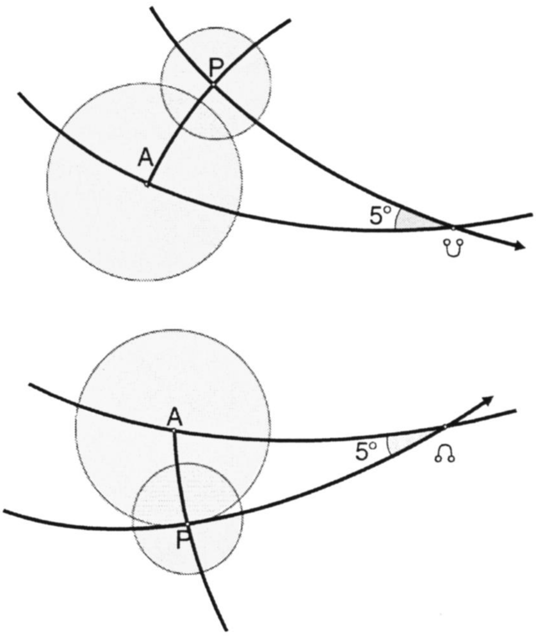


Fig. 2 The calculation of the Moon's apparent radius and the Earth's shadow

Ptolemy obtains:[12]

$$AP_1 = 0;48,30°$$

The value ΩP for the second eclipse is 7;48°. Ptolemy, presumably carrying out the same calculation, obtains:

$$AP_2 = 0;40,40°.$$

We know that AP_2 is equal to ρ_{sdw}, and thus:

$$\rho_{sdw} = 0;40,40°.$$

Finally, we know that $AP_1 = \rho_{sdw} + {}^1/_2\, \rho_{Moon}$. Therefore, ${}^1/_2\, \rho_{Moon} = 0;7,50°$ and $\rho_{Moon} = 0;15;40°$.

In this section, we first reconstructed the calculation Ptolemy used in the *Almagest* in order to calculate D_S and then we analyzed the ways in which he obtains the values involved in that calculation. For the D_L calculation, he needs the values of the moon's eccentric, epicycle and deferent radii, expressed in parts and the value of the proportion that allows converting these proportional values to absolute ones. Then, we reconstruct the calculations carried out by Ptolemy in order to obtain the two remaining values involved in the *Almagest* calculation: the apparent radii of the

[12] Ptolemy does not specify the calculation he performed to get from ☊P to AP. The results of AP are quite imprecise and there are several possibilities, which I will analyze later (Sect. 3.2.2 and ff.).

Moon's and Earth's shadow. In the next section, we will analyze the values involved in the *Hypotheses* calculation.

2.2 The calculation of D_S in the *Hypotheses*

In the *Hypotheses*, Ptolemy uses the proportions between the radii of the epicycles and deferents of the planets (of course, we are only interested in Mercury and Venus). He also used the Sun's proportions, because they are used to obtain the minimum distance (1160^{tr}) from the mean one (1210^{tr}). We need not say anything about the Moon's proportions because only its maximum distance is involved in the calculation. The calculation of the radii of the epicycles and deferent of both Mercury and Venus do not present any problem and rest on planetary observations independent of the lunar and solar ones involved in Sect. 2.1.[13] If we suppose R to be 60^{P}, the values for Venus' eccentric is $1;15^{\mathrm{P}}$, and the value for its epicycle radius is $43;10^{\mathrm{P}}$. Therefore, the maximum distance that Venus can reach is $(60^{\mathrm{P}} + 43;10^{\mathrm{P}} + 1;15^{\mathrm{P}}) = 104;25^{\mathrm{P}}$, and its minimum distance is $(60^{\mathrm{P}} - 43;10^{\mathrm{P}} - 1;15^{\mathrm{P}}) = 15;35^{\mathrm{P}}$. The proportion between these two distances is ${}^{104;25}/_{15;35}$, which Ptolemy rounds to ${}^{104}/_{16}$. In the case of Mercury, if we also assume that R is 60^{P}, the radius of the epicycle will be $22;30^{\mathrm{P}}$; the distance from the center of the Earth to the point on which the orbit of the eccentric rotates will be 6^{P} and, finally, the radius of that orbit 3^{P}. Therefore, Mercury's apogee can be calculated as $60 + 22;30 + 6 + 3 = 91;30^{\mathrm{P}}$. However, if as Ptolemy warned us, the perigee does not coincide with the position opposite to the apogee but it takes place when the planet is 120° from its apogee, it will be necessary to calculate the distance at perigee independently (IX, 9; H(2)280–282; 459–460). Ptolemy obtains: $33;4^{\mathrm{P}}$. Therefore, when Mercury's maximum distance is $91;50^{\mathrm{P}}$, the minimum distance is $33;4^{\mathrm{P}}$. Then, the proportion ${}^{91;30}/_{33;4}$. For unknown reasons, Ptolemy rounds this value to ${}^{88}/_{34}$ in the *Hypotheses*.

Let us remember, finally, that Ptolemy obtains Venus maximum distance multiplying D_L by both Venus' and Mercury's maximum-minimum distances proportions, thus obtaining a result of 1079^{tr}; while the Sun minimum distance turned out to be 1160^{tr}.

2.3 The independence of the values

With the great quantity of data and calculations, it may not be easy to appreciate at first sight the independence of the calculations made by Ptolemy and of the data he used in order to reach practically the same value: 1210^{tr}. For that reason, we will give here a brief review. In the *Almagest*, Ptolemy calculates D_S making use of three data: the Moon's maximum distance: $D_L(64;10^{\mathrm{tr}})$; the shadow's apparent radius at that distance: ($\rho_{\mathrm{sdw}} = 0;40,40°$) and, also at the same distance, the Moon's apparent radius ($\rho_{\mathrm{Moon}} = 0;15,40°$). These last two data have been obtained from a simple trigonometric calculation using the distances between the Moon's center and the nodes

[13] The reconstruction of the calculations can be found in Pedersen 1974: 295–328.

in two eclipses that had been recorded in Babylon several centuries before Ptolemy's time (nos. 4 and 5 of our Chart). The calculation of the distance between the Moon's center and the nodes is obtained knowing the time of the eclipse and applying it to the Tables, which express the results of his lunar theory.

The calculation of D_L, in turn, depends on the relative distances that Ptolemy obtains for the value of the eccentric and for the radii of the epicycle and of the deferent, and the application of a proportion ($^{59}/_{60}$) that makes those distances absolute, expressing them in terrestrial radii. This proportion is obtained by the comparison of the distances between the Moon's center and the zenith calculated both in parts and in terrestrial radii. The calculation in parts presupposes Ptolemy's lunar model. It also presupposes the calculation in terrestrial radii, but it includes in addition an observation and some other data. Indeed, the value of $39;45^{tr}$ arises from the application of the parallax (1;7°) that Ptolemy obtained comparing the observed and the calculated Moon's distance from the zenith. The calculated distance, in turn, depends on the lunar model. With all these data and calculations, Ptolemy obtained for D_L the value of $64;10^{tr}$.

In the *Hypotheses*, using that value and Mercury's and Venus' maximum and minimum distance proportions, Ptolemy obtains for the Sun a minimum distance very close to 1160^{tr}, which implies a mean distance of 1210^{tr}. In order to obtain the maximum and minimum distances of each planet, he only uses observations of these planets and, therefore, he presupposes neither the Sun's nor the Moon's distances.[14]

I think that we have sufficiently demonstrated that the methods used in order to obtain the data are absolutely independent from each other. It is not possible, therefore, to attribute the coincidence to any kind of internal interdependence. So, as most historians have supposed, the coincidences are really suspicious. And it would appear even more suspicious if we consider the sensitivity of the methods of calculations used by Ptolemy. This is the aim of the next section. Then, once we became convinced that Ptolemy had to alter or select the data, we will be ready to conjecture what was the particular way in which he did it and this is the aim of Sect. 4.

3 The accuracy of data and the sensitivity of methods of calculation

Now, we must analyze both the accuracy of the involved data and the sensitivity of the methods used. In a first very brief section, we will analyze how it is possible that, using values so erroneous, Ptolemy obtains a lunar distance which, in the syzygies, is amazingly close to the real one. We will do this in order to show that, if D_L was not a genuine datum, it was not so because it must adjust to another value and so Ptolemy did not have the freedom to modify it to achieve a specific D_S. Afterwards, we will analyze the accuracy and sensitivity of the rest of the implicated data and methods of calculation, respectively.

[14] For the calculation of the planets' position, Ptolemy actually used the Moon's theory, but he does not need to know D_L in order to carry out the calculations. Cf. Evans 1998: 250–255. The method to measure the planets' position is described by Ptolemy in VII,2; H(2)12–16; 327–329.

3.1 The *Almagest* calculation of the Moon's distance

Let us remember that, in order to obtain the absolute value of D_L, Ptolemy calculated the proportion between the absolute and the relative distance when the Moon was near the quadratures. Let us also remember that for that moment he obtains a distance of $39;45^{tr}$, a value which is well below the real one but that, on the other hand, allows to locate the Moon, in the syzygies, at a distance reasonably closer to the real one. The problem, as we have already said, lies fundamentally in that Ptolemy's lunar model placed the Moon much nearer to the Earth in the quadratures than in the syzygies. We will not analyze here the great quantity of errors involved in the values used by Ptolemy to calculate the parallax. Let us only say that the Moon, at that moment, was at $60;24^{tr}$ and not at $39;45^{tr}$.[15] However, the errors which make the Moon appear to be so close to the Earth are compensated with those of Ptolemy's model, which supposes that the Moon is indeed so close to the Earth; that way the result may get close to the correct one in the syzygies. As Toomer writes ([1984] 1998: 251, note 49), there are no accidents at all: Ptolemy knew approximately which parallax there should be in the eclipses and he picked an observation which produced that result. Besides, there are reasons to think that the value that Ptolemy finally obtains for the Moon's mean distance at the syzygies (59^{tr}) coincides closely with the value which Hipparchus had already determined centuries earlier (Toomer 1974).

3.2 The hypersensitivity of the *Almagest*'s calculation

The two calculations required to obtain D_S are methodologically very different. The one in the *Hypotheses'* is a very simple and direct calculation and all the involved parameters have, somehow, an independent corroboration. In fact, those proportions followed necessarily the distance of the eccentric and the radii of the epicycle and of the deferent. But these values cannot be altered without loss of precision in the prediction of the longitude of the planets.[16] The lunar distance, the other parameter which plays a role in the calculation, is closer to the real one and, if what we have suggested is true, Ptolemy tried to adjust it to Hipparchus' value, and, therefore, he was not able to modify it with the purpose of achieving a specific D_S.

The *Almagest's* calculation, on the other hand, is much more indirect and some of its values do not imply any consequences for the rest of the theory. Putting D_L aside again, the data obtained by the eclipses remain. Of course, in principle, if the times and magnitudes of the eclipses were correct, the obtained data could not be modified because the Tables would have to be modified too, and this would affect the precision in the prediction of the lunar longitude; but a change in times or magnitudes would not affect the whole Ptolemaic building, with the sole exception of D_S.

[15] Detailed analysis of the errors can be found in Neugebauer 1975: 102–103, Pedersen 1974: 206, Newton 1977: 182–191.

[16] It is important to remember, however, that Ptolemy's model for Mercury does not predict longitudes very accurately, so the effect of modest changes to Mercury's parameters, such as the ones mentioned between the *Almagest* and the *Planetary Hypothesis*, are not necessarily significant, and it is hard to tell the extent to which Ptolemy knew this. I thank Alexander Jones this comment.

Apart from not being fundamental for Ptolemy's model, the *Almagest* method possesses a second characteristic, which is very important to emphasize. It is a hypersensitive method: the value of D_S is very sensitive to small changes in the involved values (the magnitude of the eclipses and the values of AP, that is the line joining the center of the Earth shadow and the center of the moon). Let us first analyze the latter.

3.2.1 The variation of AP

Let us leave invariable the Moon's distance and the apparent radius of the Earth's shadow in order to analyze the variations that would follow the alteration of AP's value in the first eclipse (0;48, 30°). The equation which expresses Ptolemy's method is:

$$D_S = \frac{D_L}{\sin(\rho_{\text{Moon}}) \cdot D_L + \sin(\rho_{\text{sdw}}) \cdot D_L - 1}$$

where $\sin(\rho_{\text{sdw}}) \cdot D_L$ is, as we have seen, the absolute radius of the Earth's shadow at that distance, and $\sin(\rho_{\text{Moon}}) \cdot D_L$ is the Moon's absolute radius. In fact, Ptolemy does not obtain the shadow's absolute radius from $\rho_{\text{sdw}}(0;40,40°)$ in the calculation, but rather calculates it multiplying the lunar absolute radius ($0;17,33^{\text{tr}}$) by the proportion he has found between the apparent radii ($\rho_{\text{sdw}}/\rho_{\text{Moon}} = 2;36$). This allows him to obtain a value for the shadow's absolute radius of $0;45,38^{\text{tr}}$ when $\sin(\rho_{\text{sdw}}).\text{D}_\text{L}$ is, in fact, equal to $0;45,33^{\text{tr}}$. And this is so because, as we have seen, the proportion between the apparent radii was not exactly 2;36 but a little smaller. Besides, as we have already said, the supposition that the proportion between the apparent radii also remains in the absolute radii is an approximation.[17]

In order to be even more accurate regarding Ptolemy's calculation, we should rewrite the equation in the following way, where the absolute radius of the shadow depends on the proportion ($\rho_{\text{sdw}}/\rho_{\text{Moon}}$) and not simply on ρ_{sdw}:

$$D_S = \frac{D_L}{\sin(\rho_{\text{Moon}}) \cdot D_L + \sin(\rho_{\text{Moon}}) \cdot D_L \cdot \left(\frac{\rho_{\text{sdw}}}{\rho_{\text{Moon}}}\right) - 1}$$

the proportion between the apparent radii ($\rho_{\text{sdw}}/\rho_{\text{Moon}}$) in this case being 2;36.[18]

[17] Swerdlow 1969: 63.

[18] Hartner (1980: 20) seems not to keep this in mind when he establishes as the equation of D_S as a function of ρ_{Moon} the following one: $D = d/3.6 \sin\alpha - 1$; being $\text{D} = D_S$, $\text{d} = D_L$ and $\alpha = \rho_{\text{Moon}}$. The problem with this equation is that even if it allows us to change ρ_{Moon}, it keeps the radii's proportion constant. Therefore, the equation would only be useful, if as ρ_{Moon} grows or diminishes, ρ_{sdw} would also grow or diminish proportionally. This is the reason why the sensitivity in Hartner's equation is quite bigger than in ours (Cfr. Hartner 1980: 25). But as the value of shadow absolute radius depends on one eclipse and the value Moon absolute radius on both, Hartner's equation is not useful to see how sensitive the method is to the alteration of AP. Hartner's equation is a particular case of my own, when ($\rho_{\text{sdw}}/\rho_{\text{Moon}}$) = 2;36. Swerdlow treats the sensitivity better (1969): 63–69. There, he expresses my equation, in a more elegant way, as $D_S = D_L/D_L \cdot \sin(\rho_{\text{Moon}}) - 1$.

If we perform the calculation with the new value for the absolute radius of the shadow, the Sun would be at 1246.89^{tr}, with a difference of about 37^{tr}. The difference, which is not really big, is due to the aforementioned approximations.

Let us briefly analyze the equation. If the denominator $(\sin(\rho_{\text{Moon}}) \cdot D_L + \sin(\rho_{\text{sdw}}) \cdot D_L - 1)$ is negative, the result will be absurd, because it will yield a negative distance for D_S, which would not have any physical sense. The limit beyond which the denominator is negative can be expressed in the following way, s being the shadow absolute radius:

$$\rho_{\text{Moon}} < \sin^{-1}\left(D_L^{-1} - \sin(s)\right),$$

which in our case has a value of $\rho_{\text{Moon}} < 0;12,54$. If the Moon's apparent radius is smaller, D_S will be negative. This implies that the AP of the first eclipse cannot be smaller than $0;47,7°$, since that AP was equal to

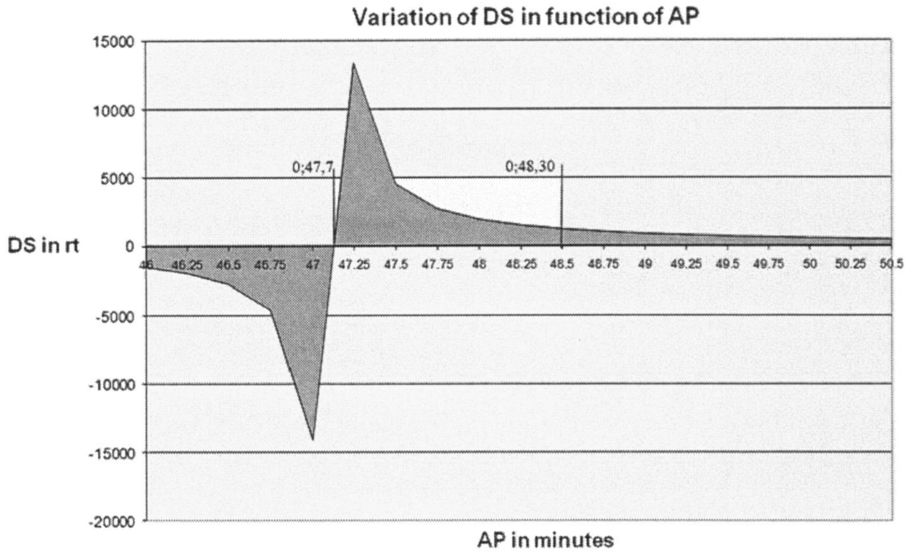
$$\rho_{\text{sdw}} + \rho_{\text{Moon}/2}.$$

A small difference, just a little more than 1 min $(0;1,23°)$, in AP would produce a disaster. The closer we come to $0;47,7°$, the more significantly D_S grows. A minute of difference (0;47,30), for example, would make $D_S = 4540^{tr}$ (see Chart A and graphic I).

On the other hand, as the denominator grows, D_S evidently diminishes. A difference of one minute equals to more than 540^{tr}. Hence, it is obvious that the method used by Ptolemy is extremely sensitive to the variations of AP.

As a conclusion, we can affirm that an error of one minute would change the obtained results so much that any coincidence with the calculation made in the *Hypotheses*

Chart A		
AP_1		D_S
′	″	tr
47	5	-45083.66
47	10	38022.75
47	15	13372.37
47	30	4540.83
48	0	1956.52
48	30	1246.89
49	0	915.01
49	30	722.66
50	0	597.14
50	30	508.76



would be impossible. A similar calculation could be made with the other two variables, D_L and ρ_{sdw}.[19]

In principle, this sensitivity would not stand against Ptolemy's method; on the contrary, it would increase the surprising character of the coincidences. Indeed, if the method of calculation is very sensitive, the employed values have to be very precise. So the question is: did the values used by Ptolemy have the precision his method demands? Let us see.

3.2.2 The calculation of AP

As we have seen, the values of AP_1 (0;48,30°) and AP_2 (0;40,40°) depend on an in principle very simple trigonometric calculation, which uses an angle of 5° (the inclination of the lunar orbit regarding the ecliptic) and the adjacent side to that angle, 0;9,20° and 0;7,48°, respectively.

Now, the way in which Ptolemy goes from ☊P, or ☋P, to AP is not clear at all. Ptolemy does not specify the calculation he has carried out and until now no method yielding exactly the results obtained in the *Almagest* had been found.

Neugebauer dedicated large attention to this problem in 1975: 106–108. There, he reconstructs the values that would be obtained by 4 different methods of calculation. Neugebauer prefers the second calculation, which is the one we used before. Toomer, on the other hand, proposes something different.[20] The differences between the results of the diverse methods of calculation and Ptolemy's values produce differences in D_S that oscillate between 152.19^{tr} and 419.90^{tr}. These are considerable differences and could put into question the coincidence between the D_S calculated here and that of the

[19] Swerdlow (1969: 63–69) has done this in a detailed way.

[20] Toomer [1984] 1998: 254, note 61.

Hypotheses. However, I believe that there is a method which adjusts itself even better to Ptolemy's values and in which the difference in D_S is no longer significant.

The method of calculation I suggest is the following. We must first remember the spherical trigonometry relationship which states that $\sin\alpha = \sin$ a /sin c, where a is AP, α is 5° and c is unknown. We know that a, the value which we want to discover, is equal to $\sin^{-1}(\sin\alpha \cdot \sin c)$. We do not know the value of c, but Ptolemy could suppose, as many times before, that c is similar to b and we know the value of b, because it is NP. Hence, the approximation would be a$=\sin^{-1}(\sin\alpha \cdot \sin b)$, i.e. AP $= \sin^{-1}(\sin 5 \cdot \sin NP)$. With this method, we obtain $\mathrm{AP}_1 = 0;48,35,35$ and $\mathrm{AP}_2 = 0;40,39,50$.[21] The total difference barely reaches 6 s, and is much closer to Ptolemy's values than the best of the other proposals (Toomer's one) which has a difference of 26 s.

With this method of calculation, the difference in seconds is barely 6. We have seen, however, that values that vary by even a minute imply important changes in D_S, but 6 s only produces a difference in D_S of 84.33^{tr} from Ptolemy's value. This difference is not significant at all. It is also interesting to notice that the result (1162.55^{tr}) gives a very close value to the Sun's minimum distance (1160^{tr}) and so, if Ptolemy wanted to alter the data so that they coincided with those of the *Hypotheses*, he could have made the calculation correctly and simply say that the D_S calculated in the *Almagest* was the minimum and not the mean solar distance. However, the inaccuracy of the calculations could be in that of ☊P or of ☋P, and not in the calculation of AP from ☊P or ☋P. We will analyze this, examining the calculations that Ptolemy should have carried out according to his own Tables.

3.2.3 The ☋P or ☊P calculation (re-calculating the values in those eclipses)

I re-calculate all the values following Ptolemy's Tables and obtain that if Ptolemy would have been more careful in his calculations of the first eclipse, maintaining the values of the second one, AP_1 would have been 0;48,48 and D_S, 1024.03^{tr}, with a little more than 200^{tr} of difference with the correct value.[22] The calculations of the second eclipse are much more accurate with a difference of only 1' in AP value (AP_2 would have been 0;40,41°, implying a difference of only 8^{tr}. So, if we make the calculations with the correct values ($\mathrm{AP}_1 = 0;48,48°$ and $\mathrm{AP}_2 = 0;40,41°$), the Sun would be at 1029.14^{tr}, with a difference of barely more than 200^{tr}. The difference is considerable, but not abysmal. We should notice, though, that the greatest part of the difference comes from eclipse n° 4, whose calculations are considerably worse than those of eclipse n° 5. In fact, we could say that there is no difference at all in eclipse n° 5, because Ptolemy rounds the numbers in minutes, obviating the seconds, and so, the value 7;48,17° for the distance from the node could be perfectly rounded to 7;48°, and in this case, AP_2 would be 0;40,40°, as Ptolemy claimed.

[21] As we will see later on (Sect. 4.3.2 and ff.), Ptolemy uses two other eclipses in order to find out the radii of the Moon and of the Earth's shadow at the Moon's minimum distance (eclipses nos. 14 and 15 of our Chart), obtaining the values ☊$\mathrm{P}_{14} = 0{:}8.20$ and $\mathrm{AP}_{14} = 0;43,20°$, and ☋$\mathrm{P}_{15} = 0;10,36$ and $\mathrm{AP}_{15} = 0;54,50°$. Keeping also these results in mind, our proposal seems to be the most convincing.

[22] It is important to remember that the calculations are performed taking $1246,89^{\mathrm{tr}}$ as D_S.

Summarizing: we have seen that the method used by Ptolemy was extremely sensitive to AP changes. The precision of the data he used is satisfactory in the case of the calculation of AP, although it is much more unsatisfactory for ☊P or ☋P, particularly regarding eclipse n° 4. However, we can say in general that the data passed the test—they are as precise as needed to match the method's sensitivity. This adds even more to the surprising character of the coincidence.[23]

3.2.4 *The variation of magnitude*

Let us now analyze how sensitive is D_S to the variation in magnitude of the eclipses. Let us consider all values constant, with the exception of the magnitude of the first eclipse. The results presented in Chart B are more than eloquent.[24] Since the magnitude of eclipses is given in whole numbers, an eclipse with a magnitude a bit bigger than 2.5 or a bit smaller than 3.5 would be rounded to 3. If this is so, the Sun's distance included in magnitude 3 would range from 3789.93^{tr} to 666.61^{tr}, depending on the exact value of the magnitude. The difference between the extremes (3123.32^{tr}) surpasses more than twice the mean distance that Ptolemy proposes for the Sun. This is so, assuming that the magnitude of the eclipse has been measured correctly. If it had been, for example, 2.4, D_S would have been $24.483.19^{\text{tr}}$, but if it had been 2.3, D_S would have gone back to $-16.402.64^{\text{tr}}$.[25] Moreover, if we consider that the lunar diameter is, like Ptolemy supposes, 0;31,20°, the difference between one magnitude and the following one is hardly 0;2,38°. It is really difficult to think that the person that estimated the magnitude of the eclipse, between 600 and 700 years before Ptolemy, had a precision of 2 min of arc.

The relative success that the precision of the methods used by Ptolemy to calculate AP have shown has no place here. Therefore, although the method to calculate the distance of the Sun using lunar eclipse magnitudes is elegant and ingenious, it is extremely unsatisfactory.[26]

In order to exemplify this, let us see what would have happened if Ptolemy had made the calculations with other eclipses.

3.2.5 *The same calculations in other eclipses*

In the *Almagest*, Ptolemy mentioned 19 lunar eclipses (see the Chart). For now, we are only interested in those eclipses in which the Moon was close to its maximum distance. We find 6 eclipses of this kind. Two of them are numbers 4 and 5, which we

[23] Of course the times and magnitudes of the eclipses proposed by Ptolemy could also be compared with the times and magnitudes that the current theories offer us. Newton 1977: 194–196, Steele 2000 and Britton 1992 have done this.

[24] The formula derived from the analysis of Ptolemy's graphics is very simple: $AP = \rho_{\text{sdw}} + [(6 - M)/6]\rho_{\text{Moon}}$. From this it follows that $\rho_{\text{Moon}}. = [6(\text{AP} - \rho_{\text{sdw}})]/(6 - \text{M})$.

[25] According to current calculations, the magnitude was, in fact, of 2.1 and so D_S would be $-3999, 33^{rt}$ or, $-2964, 65^{rt}$ if it is rounded in 2. See Britton 1992: 53 and Neugebauer 1975: 108.

[26] A parallel analysis leads Newton (1977: 181–182, 202–204) to assert that the method proposed by Ptolemy is not as good as the previous ones. Also Toomer [1984] 1998: 254 note 62 and Neugebauer 1975: 106–108.

Chart B			
Mag	Lunar radius		D_S in rt
	'	''	
1	9	24	-979.54
1.5	10	26	-1388.29
2	11	45	-2964.65
2.1	12	3	-3999.33
2.2	12	22	-6332
2.3	12	42	-16402.64
2.4	13	3	24483.19
2.5	13	25	6779.54
2.6	13	49	3789.93
2.7	14	14	2597
2.8	14	41	1938.14
2.9	15	9	1534.44
3	15	40	1246.89
3.1	16	12	1044.79
3.2	16	47	887.46
3.3	17	24	765.58
3.4	18	4	666.61
3.5	18	48	583.62
3.6	19	35	515.12
3.7	20	26	447.99
3.8	21	21	407.3
3.9	22	22	363.52
4	23	30	324.61
4.5	31	20	186.6
5	47	0	100.85
5.5	94	0	42.4

have already analyzed. In number 9, Ptolemy is not precise regarding the magnitude, and he only says that it is "partial." Eclipse number 17 is total, and so is not useful for our calculations.[27] Hence, only two remain: number 2 on -719 March 8 and number 6 on -519 November 19, both recorded in Babylon. It is interesting that the magnitude of both eclipses is equal to that of the first of the eclipses Ptolemy uses in his calculation (n° 4). One of them is earlier in time and the other later, but both are mentioned in the *Almagest* before the one Ptolemy actually used. In fact, the eclipse no 2 was used two times in the *Almagest* (H303 and H332), so there is no reason not to use it one more time. This means that, in combination with the second eclipse (n° 5), instead of introducing a new one, Ptolemy could have used any of these two to calculate the Sun's distance. And there are no apparent reasons for him not to do so, since regarding time, one is earlier and the other later; regarding place, all of them were observed in Babylon; regarding the simplicity of the calculations, all three have the same magnitude—half the lunar radius; and regarding their vicinity to the Moon's apogee, both

[27] When the eclipse is total in the moment of its maximum occultation, the Moon is completely inside the shadow, and so, in order to obtain the shadow's or the Moon's exact radii, it will not be useful to calculate the distance AP (this could be useful for calculating the minimum shadow's radii, but we will not do this in order not to complicate the argument even more).

Chart C

Lunar Eclipses	-719 March 8 (n°2)			-501 November 19 (n°6)		
	°	'	"	°	'	"
Solar mean longitude	341	24	0	233	54	30
Solar true longitude	343	45	3	233	23	21
Lunar mean longitude	164	44	33	53	39	2
Lunar anomaly	12	24	23	2	44	17
Lunar latitude	280	33	30	80	36	39
Lunar elongation	183	20	33	179	44	32
Lunar true longitude	163	45	35	53	25	49
Lunar increment	279	34	32	80	23	26
Distance from node	9	34	32	9	36	34
AP calculation	0	49	51	0	50	1

are nearer to the maximum distance than the one that in fact Ptolemy used. Let us see what would have happened if Ptolemy had used them (see chart C).

In the first eclipse, the calculation of AP is $AP_{ecl.2} = 0;49,51°$, and in the second, $AP_{ecl.6} = 0;50,01°$. With these values, supposing 0;40,40° for the other eclipse, and making the calculation with eclipse 2, D_S would be 629.96^{tr} and with eclipse 6, 597.14^{tr}.[28] It is interesting to note that both values are consistent with each other, with a difference of only 10 seconds, but both differ significantly with the value of eclipse n° 4 (0;48,30°), which Ptolemy actually uses (by approximately 0;1,30°). We can conclude, therefore that the method proposed by Ptolemy is so sensitive that, had he not applied it to the eclipses that in fact he used, the result would have been completely different.

We started this section by noting that, even if the calculation of the Moon's distance is very inaccurate, the value obtained is very close the correct one. Following Toomer, we have supposed that Ptolemy tried to adjust the value to that of Hipparchus. But the point was that, because of that, Ptolemy did not have the freedom to adjust D_L in order to obtain some specific value for D_S. After that, we analyzed the sensitivity of *Almagest* calculation to variation of AP and we proved that it is extremely sensitive. Then we described the calculation of AP from ☊P or ☋P and its different options. Subsequently, we reconstructed the calculation of ☊P and ☋P from the lunar eclipses used by Ptolemy showing that he was accurate enough. Then we analyzed the sensitivity of the *Almagest* method to variations in the magnitudes of the eclipses and

[28] The small difference that we find between Ptolemy's calculations and ours in the other two eclipses allows us to assume that he would not obtain values very different from the ones we have found. Some controls can also be made. Ptolemy says about eclipse n° 2 that the Sun's true longitude is 343;45°, and our value is 343;45,3°. Concerning eclipse n° 6, Ptolemy affirms that the lunar anomaly is 2;44° and our value is 2;44,17°, and he claims that the prosthaphairesis of longitude is 0;13° and our value is 0;13,13°. In all these calculation, I supposed, as presumably Ptolemy did, that true anomaly is equal to mean anomaly. If I introduce the difference between the anomalies, AP_6 would be almost the same (0;49,59°) but AP_2 would be 0;45,46° and, with this value and 0;40,40°, D_S would be $-1268,74^{tr}$.

we finally conclude, reconstructing the same calculation but with other eclipses used in the *Almagest*, that the result would have been completely different if he had not applied it to the eclipses that in fact he used. In the next section, we will try to explain why Ptolemy chose what he actually chose.

4 The hypothesis of the alteration or selection of data

I believe there is enough evidence to assert that, in fact, Ptolemy deliberately altered or selected the data so that his results would be consistent with a predetermined value. In this section I will propose some possible ways in which Ptolemy could have done this.

My conjecture is the following. Let us assume that Ptolemy decided to calculate D_S using eclipse n° 2 or n° 6. When carrying out the calculations he would have obtained a distance of about 600^{tr} and, after carefully revising them, he would have concluded that, evidently, the result was correct. Possibly, Ptolemy suspects that the correct result would be greater than what he actually obtains, maybe because he still has some confidence in Aristarchus' proportion. In any event, it seems very unlikely that Ptolemy would have calculated D_S using only two eclipses without verifying the result with others. And taking into account the method's hypersensitivity, it is almost impossible that he would have obtained consistent results using different pairs of eclipses. At that moment Ptolemy would have looked for some independent way of obtaining the value of D_S with the purpose of being able to choose the data, he would introduce in his method of calculation in a better way. Ptolemy was not able to measure the solar parallax because it is very small and so the natural candidate (probably the only one) is the one that he uses in the *Hypotheses* to calculate the planets' absolute distances. Making a calculation similar to the one of the *Hypotheses*, Ptolemy would have obtained a value close to 1210^{tr} for the Sun's distance and he would then search for the eclipses that gave him the correct value. That Ptolemy had the planets' proportions before calculating D_S in the *Almagest* is beyond any doubt because they are in the *Canobic Inscription*, which precedes the *Almagest*. The eccentricities and the radii of the epicycles of the planets are, with small exceptions for Saturn and Mercury, the same as in the *Almagest*. There is no scientific fraudulence in doing this. Based on another part of his theory, and aware of the method's sensitivity, he simply chose the data that suited him most.[29] Of course, there would have been dishonest if, after calculating the data, he had emphasized the coincidence as a test of his theory, but Ptolemy never does this; on the contrary, he recognizes the discrepancy between the calculations of the *Almagest* and of the *Hypothesis* as a difficulty that must be answered, not as a coincidence that corroborates the theory. Moreover, the fact that he did not explain his strategy used to choose the eclipse is understandable because throughout the *Almagest*

[29] This strategy of choosing the data so that they agree with previous values is frequent in Ptolemy. We have already seen this in the case of the Moon's mean distance at the syzygies which coincides with the Hipparchian value. There is also an extensive discussion about the equinox's precession in which it seems that Ptolemy also adjusted or selected the values so that they would coincide with Hipparchus'. See Toomer 1974: 131, note 25. For this discussion see, for example, Newton 1977 and 1979; Grasshoff 1990; Evans 1987 and 1998: 264–274; Gingerich 1980 and 1981; Dreyer 1917 and 1918; Swerdlow 1992.

Ptolemy takes great care to not use data that he has not explained or demonstrated yet. At this place in the *Almagest*, he has not developed the model for the planets yet, and so, Ptolemy cannot logically introduce their maximum and minimum distances.[30]

4.1 Rounding values and the exact coincidence

Now, if all this is so, how can there be a difference of 89^{tr} between the result of the calculation of the *Almagest*, and the same calculation, repeated in the *Hypotheses*? Ptolemy accepts that there is a discrepancy which he cannot explain, but recognizes that he has necessarily reached those values. Many historians have highlighted that it is extremely curious that Ptolemy does not realize that, had he not rounded the numbers, the discrepancy could be explained.[31] How is it possible that Ptolemy, clearly an accomplished mathematician, did not realize that the discrepancy was due to working with rounded numbers? This question becomes much more pressing if one accepts my hypothesis, because if Ptolemy had chosen the values of the eclipses so that they would fit with those of the distances of the planets, he would have already known that the results would be the same, not only because he trusted in the truth of his theory, but also because Ptolemy himself would have carried out the calculation previously. In addition, even if it had not rounded the numbers, the discrepancy would have continued to exist, although now it would be only of 29^{tr}. If he calculated D_S in the *Almagest*, presupposing the values that he would then use in the *Hypotheses*, how is it possible that later on they do not fit perfectly or, at least, with a much closer precision?

I will try to answer both questions. Most historians have drawn attention to the fact that Ptolemy rounded the numbers and also that, in the case of Mercury, he rounded them wrongly, going from ${}^{91;30}/_{33;4}$ to ${}^{88}/_{34}$. Moreover, many historians calculate the minimum D_S without rounding and with the correct values extracted from the *Almagest*; as we have seen, they obtained 1189^{tr}. Until now, nobody has made the calculation rounding with the correct numbers. However, let us suppose that Ptolemy did not worry about rounding because he indeed knew that the rounded numbers would yield the correct value. However, if this is true, then the result with rounded numbers should be closer to 1160 than the one without rounding. Such is indeed the case if we correct Ptolemy's Mercury rounding error. If we round correctly ${}^{91;30}/_{33;4}$ to ${}^{92}/_{33}$—not to ${}^{88}/_{34}$—the value for the Sun's minimum distance is astonishing: $64.\ {}^{92}/_{33}.\ {}^{104}/_{16} = 1159.76^{tr}$ or, rounded, exactly 1160^{tr}! We have obtained the correct value, with a difference smaller than a quarter of a terrestrial radius.[32]

---

[30] Swerdlow (1968: 102) supposes that at the moment he wrote the *Almagest*, Ptolemy had not developed his theory of planetary distances yet, because he says in the *Almagest* that the planets have an imperceptible parallax, but in the *Hypotheses* he says that Mercury's and Venus' parallaxes are not insignificant. However, after calculating D_S and D_L, Ptolemy could have very well known that Venus and Mercury—which would surely be between the Moon and the Sun—would exhibit a parallax between the lunar and solar parallaxes, and so that they would not be imperceptible. Therefore, we must conclude that the reason is—as Swerdlow also proposes, although he then prefers the other hypothesis—that Ptolemy says that in the *Almagest* because the parallaxes are irrelevant for the calculation of the longitudes of the planets.

[31] van Helden 1986: 23; Swerdlow 1969: 127; Newton 1977: 71–72; Pérez Sedeño (1987: 38), in the only Spanish translation of the *Hypotheses*, also notices it.

[32] The 91;30 could also be rounded to 91. With those values, D_S would reach the 1147, 16^{tr}.

This coincidence is too suspicious not to be considered. If it looked extremely suspicious until now that Ptolemy had achieved a correspondence between 89^{tr} and 14^{tr}, it would look even more suspicious now that the difference is hardly 0.24^{tr}, or simply that there is no difference at all, if we consider that Ptolemy would round the number. How should we interpret this? When Ptolemy calculated an approximate value for D_S in order to choose the correct eclipse, he simply rounded the numbers correctly and he therefore obtained a minimum distance of the Sun of 1160^{tr} and a mean one of 1210^{tr}. These are the values he used in order to choose the eclipse that would give a value of 0;48,30° for AP. Then, when he writes the *Hypotheses*, Ptolemy calculates the planetary distances rounding again—because he had made the previous calculation rounding the numbers—but he makes a mistake when rounding the values of Mercury.

We have answered, thus, both questions: there is an absolute consistency between the values obtained in the *Almagest* and the (correctly) rounded values of the *Hypotheses*. On the other hand, Ptolemy did not worry about the rounding because he had made the calculations by rounding. If we bear in mind that Ptolemy was looking for an approximate value of D_S to find the appropriate eclipse, it seems most natural that he made the calculation with rounded numbers.[33]

4.2 Objections to the conjecture

Our hypothesis seems to be well founded, but it still has to face a strong objection: how to explain that Ptolemy made a mistake rounding ${}^{91;30}/_{33;4}$ to ${}^{88}/_{34}$. It is a fact that he made that mistake, but it is even more difficult to explain, assuming that he knew that the calculation should give him exactly 1160^{tr}. If he already knew the result of the calculation, why, when he obtained a different result (1079^{tr}), did he not revise the calculation?

The answer is simple. Ptolemy, as we have already pointed out, changed the proportions of Mercury in the *Hypotheses*, with respect to those used in the *Almagest*, presumably due to new and better observations. Actually, he changed the epicycle radius from $22;30^P$ to $22;15^P$ and the radius of the eccentric's orbit from 3^P to $2;30^P$. Hence, the maximum distance of Mercury became $(60+3+2;30+2;30+22;15) = 90;15^P$, and the value of the perigee became $33;47^P$, which could be rounded to 34^P.[34] The new value, therefore, for Mercury, would be correctly rounded to ${}^{90}/_{34}$. In that case, the error remains only in the 88. Ptolemy knew that he had changed the values for Mercury by reducing its proportions, and therefore he expected a value for D_S smaller than the one he had previously calculated. It is true that he made a mistake by putting 88 instead of 90, but the result—a value a bit smaller than 1160—did not astonish

[33] Undoubtedly, if Ptolemy calculated an approximate D_S, we have no reason to explain why the value's coincidence is exact and not only approximate. I believe that the answer to this objection is the following: the method used by Ptolemy is so sensitive to changes that it is easier to achieve the exact value than an approximate one because, if one does not proceed with much care, the final result will be significantly different.

[34] Cf. Heiberg 1907: 87, 89 (Pérez Sedeño 1987: 66–67). The calculation of $33;47^P$ appears in Swerdlow 1968: 118. Cfr. also Goldstein 1967: 9–10; Hartner 1964:267.

him, because it would follow from the change he had introduced. With the proportion $^{90}/_{34}$, the Sun's minimum distance would be $(64 \cdot {}^{90}/_{34} \cdot {}^{104}/_{16} =)1101.17^{tr}$. With the proportion $^{88}/_{34}$, the value would be 1079^{tr}, as we have already said. However, what might have prevented Ptolemy from noticing the error is that both results are a little bit smaller than 1160^{tr}, as he expected.

I think the answer to the objection is satisfactory. However, I am aware that the reader may not be satisfied with it because he might think it difficult to suppose that, once Ptolemy had obtained the right results in the first instance, then he would have made a mistake. I suggest an alternative conjecture, which even if we suppose that Ptolemy used the nesting-spheres calculation to choose the eclipses, does not assume that he made the same calculations twice. Of course there are many ways in which Ptolemy could have made the calculations. Here I will suggest the simplest one. Ptolemy could have asked himself: which is the easiest and simplest way of obtaining approximate proportions for the planets? This would be to use a simple epicycle-deferent system (just one deferent and one epicycle, without eccentrics). In the case of inner planets, we can calculate the proportion of the epicycle taking the maximum elongation into account. The proportion of the epicycle would be equal to the sine of the maximum elongation multiplied by 60 parts. In the case of Mercury, Ptolemy knew that the maximum elongation was about 28°, and for Venus, about 47°. So Mercury's epicycle would be:

$$\sin(28) \cdot 60^{p} = 28;10^{p}$$

and Mercury's maximum distance $(60 + 28;10 =)88;10^{p}$[35] and Mercury's minimum distance $(60 - 28;10 =)31;50^{p}$.

On the other hand, Venus' epicycle would be:

$$\sin(47) \cdot 60^{p} = 43;53^{p}$$

and Venus' maximum distance $(60+43;53 =)103;53^{p}$ and Venus' minimum distance $(60 - 43;53 =)16;07^{p}$. With these values, the Sun's minimum distance would be:

$$64;10 \cdot ({}^{88;10}/_{31;50}) \cdot ({}^{103;53}/_{16;07}) = 1145;31^{p}.$$

Of course, we cannot be sure that Ptolemy used these values for the maximum elongations, but this method of calculation is not very sensitive to reasonable variations in them. For example, if we suppose, as a lower limit, 45° for Venus maximum elongation and 26° for Mercury, D_S will be $957;49^{tr}$. On the other hand, if we suppose, as the upper limit, 49° for Venus and 30° for Mercury, D_S will be $1377;4^{tr}$.

The conclusion is clear: even if, in order to obtain reliable data for the selection of eclipses, Ptolemy had not made the calculation I suggested as the first alternative, the simplest calculation—which only presupposes the maximum elongation—would have yielded a value close enough to 1160.

35 Maybe in this value (88;10) we can find an explanation of the number 88, in the wrongly rounded values of Mercury's proportion.

There is another possible objection. As explained earlier, Ptolemy could have not say anything about the method of nesting spheres when he calculated D_S in Book V, but there is another place where the silence of Ptolemy seems to be more problematic: the first chapter of Book IX where he analyzes the order of the spheres. Why didn't he mention the nesting spheres method? Here we must keep in mind the difference between establishing the order of the celestial bodies and determining their distances to the Earth. It is clear that if you know the distances, you also know the order, but it is not necessary to know the distances in order to know the order. In the IX Book of the *Almagest*, Ptolemy is talking about the order, not the distances, and he can establish the order without any reference to the distances. Moreover, because the method of the nesting spheres supposes the order in order to determine the distances, he cannot appeal to this method to establish the order. The reason for the order that he gives in the *Almagest* is that the Sun divides those planets "which reach all possible distances from the sun and those which do not do so, but always move in its vicinity" (IX,1; H(2) 207; 419–420). This reason is independent of the distances, so there is no reason to mention distances in this part of the *Almagest*. Another problem with this chapter is that Ptolemy asserts that "none of the stars [the five planets and the fixed stars] has a noticeable parallax" and that this "is the only phenomenon from which the distances can be derived" (IX,1; H(2) 207; 419), and, if he already had in mind the nesting sphere method, he had been realized that at least Mercury must have a noticeable parallax (in fact, when it is in perigee, Mercury must have the same parallax of the Moon at its maximum distance). But Ptolemy says exactly the same thing, that is, that they have no noticeable parallax, also in the Hypotheses, where he obviously has in mind the nesting spheres method. In fact, in the Hypotheses (I-II, 2; Goldstein 1967: 6) he says that "no phenomenon allows us to fix [planets'] parallax with certainty". Finally, as a general answer to both silence objections we might mention that there are several places in Ptolemy's work, particularly in the *Almagest*, where his silence is baffling, e.g. in the apparent diameter of the Moon at quadrature.

4.3 Supporting facts

In this section, we will analyze two facts that increase the plausibility of our explanation. The first one is related to the calculation of the planets' absolute radii, particularly one mistake made by Ptolemy in the calculation of that of Venus, and the second one has to do with the calculation of the Earth's shadow and the Moon's apparent radii at the moon minimum distance.

4.3.1 Another curious mistake: Venus' absolute radius

In the *Hypotheses* (I-II, 5; Goldstein 1967: 8–9), after calculating the absolute distance of each planet, Ptolemy deduces their absolute radii and then their volumes. In doing so, he uses certain values of the planets' apparent diameters relative to the Sun's. The calculation is very simple: it is enough to multiply the sine of the apparent radius by

Chart D							
	Distance			Diameter			Volume
	Maximum	Minimum	Mean	Relative-to-the-Sun	Diam. By distance	Absolute diameter	
Moon	64	33	48	$1\,^1/_3$	64	$1/4 + {}^1/_{24}$	$^1/_{40}$
Mercury	166	64	115	$^1/_{15}$	8	$^1/_{27}$	. $^1/_{19683}$
Venus	1079	166	622.5	$^1/_{10}$	62	$1/4 + {}^1/_{20}$	$^1/_{44}$

the planet's distance to obtain the absolute radius.[36] In order to do this, it is necessary to convert the apparent radius (relative to the Sun) to an absolute apparent radius. This can be done by multiplying the apparent radius by a factor which Ptolemy obtains by dividing the Sun's apparent radius and the Sun's distance, both expressed in terrestrial radii: $^{5;30}/_{1210}$. Ptolemy's result is $^1/_{220}$. The calculations which Ptolemy in fact performs are a little longer, but he arrives at the same result.

Ptolemy claims that the apparent diameters have been measured at the planets' mean distance. The Moon's mean distance is [(34 + 64)/2] equal to 48;30 and Ptolemy rounds it to 48; that of Mercury is [(64 + 166)/2] = 115; and Venus's is [(166 + 1079)/2] = 622;30 (Ptolemy does not round it). Of course, Ptolemy continues with the calculations of the outer planets, but we are not interested in them now.

Afterwards, Ptolemy multiplies these distances by the apparent diameters. The Moon's is $1\ ^1/_3$ times the Sun's;[37] Mercury's one, $^1/_{15}$, and Venus's one $^1/_{10}$. The results are, for the Moon: 64, for Mercury: 7.7 (which Ptolemy rounds to 8), and for Venus: 62.25 (which he rounds to 62).

These values are then multiplied by the scale factor ($^1/_{220}$), resulting in the absolute diameters, expressed in terrestrial diameters. For the Moon, Ptolemy obtains: $1/4 + {}^1/_{24}$, for Mercury: $^1/_{27}$, and for Venus: $1/4 + {}^1/_{20}$. Then, he calculates the volumes, by cubing the diameters. Ptolemy obtains for the Moon a volume of $^1/_{40}$, for Mercury $^1/_{19683}$, and for Venus $^1/_{44}$ (see Chart D).

However, there is a problem with Venus' calculations. The real diameter that Ptolemy records $(1/4 + {}^1/_{20})$ does not follow from the calculations. Indeed, $^{62}/_{220}$ is 0.281, which is much closer to $1/4 + {}^1/_{30}$ [0.283] than to $1/4 + {}^1/_{20}$ [0.3]. The calculation of Venus' volume, on the other hand, is performed supposing the value $1/4 + {}^1/_{30}$, since $(1/4 + {}^1/_{30})^3 = {}^1/_{44}$, while $(1/4 + {}^1/_{20})^3 = {}^1/_{37}$. It could be simply considered a transcription error, as Goldstein (1969:12) supposes in his edition of the

36 Actually, Ptolemy did not multiply by the sine of the radius, but directly the radius, in a typical approximation for him. Working with such small angles the difference is insignificant.

37 In this case the Moon's diameter is bigger than Sun's. This is consistent with having supposed that the Moon and the Sun have the same radius when the Moon is at its maximum distance; although, of course, it does not coincide at all with the observations. Goldstein (1967: 11) considers this a sign that Ptolemy took the exaggerated variations of the Moon's distance implied by his model seriously. A more plausible answer is that, if he had not assumed that the Moon had a diameter of $1\ ^1/_3$ he would have never got a radius of $1/4 + {}^1/_{24} (= 0;17,30°)$, which coincides with the one he had calculated in the *Almagest* (0;17,33°). (In fact in the *Hypotheses*, Ptolemy calculates the diameters, but taking a terrestrial diameter as unit, while in the *Almagest* he calculates the radius, but taking the terrestrial radius as unit; thus, the values should coincide).

Chart E								
	Distance			Diameter			Volume	Order
	Maximum	Minimum	Mean	Relative-to-the-Sun	Diam. By distance	Absolute diameter		
Venus A	1079	166	622.5	$^1/_{10}$	62	$^1/_4 + ^1/_{30}$	$^1/_{44}$	L>V
Venus B						$^1/_4 + ^1/_{20}$		V>L

Hypotheses. However, even if we suppose a transcription error there is another fact which still requires an explanation. After enumerating all the volumes, Ptolemy orders the planets from the biggest to the smallest volume and he places Venus before the Moon. That is to say, he affirms that Venus' volume is bigger than the Moon's volume. The Moon has a volume of $^1/_{40}$ and so this assertion is true if we suppose the value $^1/_{37}$, but it is not if we consider $^1/_{44}$. Therefore, it cannot simply be considered a transcription error, unless we suppose another error to explain why he inverted the planets' order. Swerdlow (1969: 171–172) notices that Ptolemy made a mistake in the order of the planets, but he assumes directly the value $^1/_4 + ^1/_{30}$, mentioning Goldstein's supposed correction, although he does not link those two errors with each other.[38]

It looks as if Ptolemy was simultaneously using two calculations. In Chart E, the data shadowed are those which in fact appear in the *Hypotheses* and each line (A and B) represents each supposed calculation:

Let us suppose that the calculation of Venus B is that which Ptolemy would have made according to our hypothesis, that is, with the correct rounding. Venus' maximum distance is already known: 1160^{tr}. Then, Venus' minimum distance would be:

$$64 \cdot {}^{92}/_{33} = 178.42 \text{ (which Ptolemy would round to 178).}$$

Therefore, Venus' mean distance would not be 622.5 but [(178 + 1160)/2] = 669. With an apparent diameter of $^1/_{10}$, the multiplication of the distance by the diameter would be: 66.9, which Ptolemy naturally would round to 67. If we divide it by 220, we obtain a value of 0.3045 which, if Ptolemy rounds to two decimals (0.30), is exactly $^1/_4 + ^1/_{20}$.[39] (See Chart E').

38 Neugebauer (1975: 922) writes $^1/_4 + ^1/_{20}$ but does not mention the inconsistency in the other calculations.

39 The level of precision with which Ptolemy obtains the real diameter, however, represents an additional complication. In the case of the Moon, for example, Ptolemy obtains the value of $^1/_4 + ^1/_{24}$ from $^{64}/_{220}$. Presumably, he supposed that it was equal to $^{55}/_{220} + ^9/_{220}$, and since $^{55}/_{220}$ is $^1/_4$, and $^9/_{220}$ is quite near to $^1/_{24}$, Ptolemy asserted that the Moon's diameter was $^1/_4 + ^1/_{24}$. If we follow the same method, the result of the calculation performed with our hypothesis would be $^1/_4 + ^1/_{18}$ and not $^1/_4 + ^1/_{20}$ since $^{67}/_{220}$ is equal to $^{55}/_{220} + ^{12}/_{220}$ and $^{220}/_{12}$ is closer to 18 than to 20. However, we should remember that, according to our hypothesis, the Moon's and Mercury's values come from different calculations, presumably made in different times and contexts, and so they can have different level of accuracy. To round $^{67}/_{220}$ in $^1/_4 + ^1/_{20}$ is more than acceptable (it is, as we have seen, to round 0,304 in 0,3). If we want to have the same degree of precision, a possible explanation would be to suppose that Ptolemy mixed the minimum distance values of the calculation A with the maximum of B, which would provide a mean distance of $(166 + 1160)/2 = 663$, which Ptolemy would round in 66 and $^{66}/_{220}$ is exactly $^1/_4 + ^1/_{20}$. If one would not like to accept that Ptolemy carried out the calculations which our hypothesis poses, it could still be said that the value is a result of the average between the minimum distance of the calculation A (166) and the minimum distance of the Sun (1160) obtained in the *Almagest*, assuming—like Ptolemy could do—that

Chart E'								
	Distance			Diameter			Volume	Order
	Maximum	Minimum	Mean	Relative-to-the-Sun	Diam. By distance	Absolute diameter		
Venus A	1079	166	622.5	$^1/_{10}$	62	$^1/_4 + {^1/_{30}}$	$^1/_{44}$	L>V
Venus B	1160	178	669	$^1/_{10}$	67	$^1/_4 + {^1/_{20}}$	$^1/_{37}$	V>L

Hence, we can conclude that it is highly probable that the error of introducing the value of $^1/_4 + {^1/_{20}}$ and claiming that Venus is more voluminous than the Moon is due to the fact that Ptolemy would have taken those two data from previous calculations, in which he had not yet modified Mercury's parameters.[40] This is one more extremely eloquent sign that Ptolemy would have performed those calculations.[41]

4.3.2 The calculation of the Earth's shadow and the Moon's apparent radii at the minimum distance

As we have seen, Ptolemy could have used other eclipses but in fact he did not. We have speculated that this is due to the fact that he knew what values he should find. But, which was exactly the procedure followed by Ptolemy? Of course, we cannot know this with certainty, but we will explain a procedure that we believe is extremely plausible.

Let us first highlight certain curious data. Ptolemy calculated the apparent and absolute radii of the Earth's shadow and of the Moon at the Moon's maximum distance by

Footnote 39 continued
Venus' maximum distance should coincide with the Sun's minimum one. But again, this presupposition has no basis if our hypothesis is not accepted, namely that Ptolemy had made the calculation previously and he knew that they should coincide.

40 If Ptolemy had not rounded the mean distance of the Sun from 48;30 to 48, the result of the lunar diameter would vary very little. It would be of $^1/_4 + {^1/_{23}}$, which does not influence at all their comparison with Venus, which oscillates in both calculations between $^1/_4 + {^1/_{20}}$, and $^1/_4 + {^1/_{30}}$. In Mercury's case, the differences between the results of calculations A and B are so small that they are lost in the rounding.

41 It is interesting to notice that the discrepancy between the value of Venus' diameter and the order of the planets on one hand, and their volume and all the other values on the other, has been clearly perceived by medieval astronomers and they have tried to solve it. Some of them believed that the error was in the diameter—as Goldstein does—but several important authors believe that the error was in the volume calculated by Ptolemy, and that the diameter and the order were correct. This way, for example, Al-Fargani in chapter 22 of his *Elementa Astronomica* (Campani 1910: 148–150f, cfr. Swerdlow 1969: 174–175) dedicated to planet and star sizes, affirms that Venus' volume is $^1/_{37}$. Similarly, Thabit ibn Qurra—the probable author of the Arab translation of the *Hypotheses*, the oldest that we have—who in his *De his quae indigent expositio anequam legatum Almagesti* (Carmody (1960: 136–137, n. 42) copies the distances which appear in the *Hypotheses*, and whereas the radii do not appear, the volumes do and, in Venus' case, it is of $^1/_{37}$ and not $^1/_{44}$. Al Battani (Nallino 1899–1907(I): 123, N° 13–17, cfr. Swerdlow 1969: 179–181) claims that the volume is $^1/_{36}$, thus retaining the diameter $^1/_4 + {^1/_{20}}$. Ibn Rustah (de Goeje 1982: 20–22, cfr. Swerdlow 1969: 176–178), on the other hand, says that the volume is $^1/_{44}$, correcting $^1/_4 + {^1/_{20}}$ with $^1/_4 + {^1/_{30}}$, and the order in the planets enumeration according to their volumes. Swerdlow (1969: 176) tries an explanation of the apparent "error" in the volume appearing in Ibn Qurra's text. He says that Al-Fargani's error might have been introduced by contamination into the text and that the fact that the volume coincides with the order of the *Hypotheses*, would have added to its plausibility. However, he does not realize that it is the value that follows from $^1/_4 + {^1/_{20}}$.

Chart F												
Lunar Eclipses	-173 May 1 (n° 14)						-140 January 27 (n° 15)					
	Ptolemy			Tables			Ptolemy			Tables		
	°	′	″	°	′	″	°	′	″	°	′	″
Solar true longitude	36	15	0	36	14	29	305	8	0	305	8	17
Lunar mean longitude	217	49	0	218	1	44	125	16	0	125	16	22
Lunar anomaly	163	40	0	163	50	46	178	46	0	178	46	22
Lunar true longitude	216	16	0	216	30	20	125	8	0	125	8	57
Lunar increment	98	20	0	98	29	18	280	36	0	280	37	46
Distance from node	8	20	0	8	29	18	10	36	0	10	37	46
AP calculation	0	43	26	0	44	14	0	55	7	0	55	16

means of the two eclipses that we have already analyzed. In order to calculate when an eclipse will take place, he also needed the apparent radii of the Earth' shadow and of the Moon when the Moon is at the minimum distance and in syzygy. The calculation is simple if we already know the absolute values of the Earth's shadow and of the lunar radius, as well as the distance between the maximum and the minimum distance at syzygy. The Moon's absolute radius has been calculated as $0;17,33^{tr}$ and the shadow's as $0;45,38^{tr}$. Ptolemy established the Moon's minimum distance at syzygy as $53;50^{tr}$, therefore:

$$\text{Minimum}\rho_{\text{Moon}} = \sin^{-1}\frac{0;17,33}{53;50} = 0;18,41^{\circ}$$

$$\text{Minimum}\rho_{\text{sdw}} = \sin^{-1}\frac{0;45,38}{53;50} = 0;48,34^{\circ}$$

However, Ptolemy does not follow this simple calculation, but rather he calculates them 'empirically' using two other eclipses, which occurred near the Moon's minimum distance.

The first one, on -173 May 1, was observed in Alexandria (n° 14 of our Chart), with a magnitude of 7 digits from the north, and the second one, on -140 January 27, was observed in Rhodes, with a magnitude of 3 digits from the south (n° 15).

Concerning the first one, Ptolemy claims the true longitude of the Sun to be 36; 15°, the mean longitude of the Moon, 217;49°, and the true longitude 216; 16°, with a lunar anomaly of 163;40° and a lunar increment of 98; 20°, and hence, the moon was at 8; 20° from the node. This would yield a value of AP = 0;43, 20°. Our calculations, with that distance from the node, give: 0;43, 26°. However, had the data been more carefully calculated, the final value of AP would be 0;44; 14° (see Chart F).

Concerning the second eclipse, Ptolemy says that the true longitude of the Sun was 305; 8°, the Moon's mean longitude: 125; 16°, and the true longitude 125;8°. The lunar anomaly was 178;46°, the lunar increment 280;36°, and hence, the Moon's distance from the node 10;36°. According to Ptolemy, the value of AP was 0;54,50°. Our calculation of AP with those values gives 0;55,7° and, with a more careful calculation for all values: 0;55,16°.

Ptolemy carries out the same calculations as with the other two eclipses and obtains that the apparent lunar radius at that distance is 0;17,40°, and the shadow's apparent radius is 0;46°. Obviously, these values differ widely from the ones obtained in our previous calculation: 0;1,1° for the minimum Moon apparent radius, and 0;2,34° for that of the shadow.

This is very striking, since it represents a strong inconsistency in Ptolemy's system. From a different perspective, if we calculate the absolute radii with the values that Ptolemy obtains from these new eclipses, the Moon's value would be: $0;16,36^{\mathrm{tr}}$ and, the shadow's: $0;43,13^{\mathrm{tr}}$, a considerably different value from the previous ones ($0;17,33^{\mathrm{tr}}$ and $0;45,38^{\mathrm{tr}}$). Why didn't Ptolemy calculate the values for the radius at the minimum distance directly from the maximum, which he had already obtained? Or why didn't he correct them when he perceived the discrepancy? It is difficult to believe that Ptolemy had not realized this inconsistency.

A very curious fact can help us out. When Ptolemy obtains the two values (0;17,40° and 0;46°), he affirms, by way of corroboration, that the proportion between them is "negligibly *greater* than 2 $^3/_5$[2;36]" (VI,5; H480; 285). It is necessary to remember that, when he had made the calculation with the first two eclipses at the Moon's maximum distance, he had said that the proportion was "negligibly *less*" than the same number.

So, using the values that Ptolemy had obtained, it is easy to calculate the apparent radii of the Moon and of the Earth's shadow at the Moon's mean distance, by calculating the average of the maximum and the minimum one. Therefore:

$$\mathrm{mean}\rho_{\mathrm{Moon}} = \frac{0;17,40^\circ + 0;15,40^\circ}{2} = 0;16,40^\circ$$
$$\mathrm{mean}\rho_{\mathrm{sdw}} = \frac{0;40,40^\circ + 0;46^\circ}{2} = 0;43,20^\circ$$

However, the proportion between both values is exactly 2;36! Neither negligibly greater nor negligibly less. It seems as if Ptolemy had calculated the values at the Moon's mean distance and he had then tried to find them in the maximum and minimum distances.

Ptolemy also had to realize that the absolute radius of the Earth's shadow, contrary to the Moon's radius, should diminish as we move away from the Earth and so, although very little, the proportion should change at the maximum, mean and minimum distances, being smaller at the maximum distance and greater at the minimum one, where the shadow is bigger. But the difference had to be negligible, because, the Sun being so far, the variation in the radius of the shadow had to be practically insignificant. Assuming that the proportion was 2;36 at the mean distance, Ptolemy knew that at the maximum distance it should be negligibly less than that value, and at the minimum distance, negligible greater. This is exactly what he finds in the eclipses.

Everything leads us to suspect, then, that Ptolemy started from the mean distance, where he knew the values of the radii of the Moon and of the Earth's shadow,[42] and,

[42] From the value of the Moon's apparent radius at the Moon's mean distance, (0;16,40°) follows a diameter of 0;33,20°, which is suspiciously similar to the one which Ptolemy himself says that Hipparchus had

Chart G									
Lunar eclipses at Moon's mean distance	**-490 April 25 (7)**			**125 April 5 (16)**			**134 October 20 (18)**		
	°	′	″	°	′	″	°	′	″
Solar mean Longitude	29	10	44	11	38	39	206	42	23
Solar true Longitude	30	32	30	13	31	54	205	9	7
Lunar mean Longitud	215	40	1	188	30	37	29	50	58
Lunar anomaly	100	19	22	251	52	43	64	58	37
Lunar Latitude	84	26	17	74	32	32	100	29	3
Lunar Elongation	186	29	17	176	51	58	183	8	35
Lunar true Longitude	210	38	22	193	26	26	25	34	6
Lunar increment	79	24	38	79	28	21	96	12	11
Distance from node	10	35	2	10	31	39	6	12	11
AP Calculation	0	55	4	0	54	45	0	32	23

therefore, the proportion between them, and then calculated which values they should have at the Moon's maximum and minimum distances. Or, perhaps, in spite of the differences in the different eclipses, he had found that the proportion 2;36 remained more or less constant, and he also knew the Moon's apparent radius at the mean distance which he had presumably taken from Hipparchus.

Out of 19 lunar eclipses recorded in the *Almagest*, 8 took place when the Moon was at its mean distance, but 3 of them are total (n° 1, 12 and 13), and for 2 we have incomplete data (8 and 11); hence we only have 3 useful eclipses: the one that took place on −490 April 25 (n° 7) in Babylon, with a magnitude of 2 digits from the south; the eclipse which occurred in Alexandria on 125 April 5 (n° 16), with the same magnitude; and the one that took place on 134 October 20 (n° 18), also in Alexandria but with a magnitude of 10 digits from the south. Since eclipses n° 7 and n° 16 have the same magnitude, we should obtain the same value of AP. And, indeed, the difference between them is insignificant: $AP_7 = 0;55,4°$ and $AP_{16} = 0;54;45°$. On the other hand, AP_{18} is 0;32;23° (see Chart G).

In order to resolve the discrepancy of values between AP_7 and AP_{16}, we could choose the second one corresponding to an eclipse occurring during Ptolemy's life,[43] like AP_{18}. Hence, we would obtain the following two equations:

Footnote 42 continued
found: 0;33,14°. Actually, Ptolemy says (IV,9; H327; 205) that the Moon's radius at its mean distance was, for Hipparchus, of approximately $^{360}/_{650}$, which gives a value of 0;33,14°. In turn, the 0;33,20° value as lunar diameter appears in the *Canobic Inscription*. Swerdlow 1968: 74–77, suspects that those results come from Hipparchus and not from Ptolemy, then we could suppose that 0;33,20° was Hipparchus' value. However, the problem is extremely complex because of a lot of inconsistencies appearing in the *Canobic Inscription*.

43 Contrary to eclipse n° 18, which he says to have observed carefully himself (IV, 6; H314; 198), it does not seem probable that Ptolemy performed this observation himself, but that Theon did (see Toomer [1984] 1998: 206, note 54). Curiously, Britton first asserts that eclipse n° 16 was probably observed by Ptolemy because "the time reported for this eclipse is given in equinoctial hours relative to midnight" (Britton 1992: 51, note 7) but then, citing Toomer's note 54, says the opposite (page 69).

$$0;54,45^\circ = \rho_{\mathrm{sdw}} + \frac{2}{3}\rho_{\mathrm{Moon}}$$
$$0;32,23^\circ = \rho_{\mathrm{sdw}} - \frac{2}{3}\rho_{\mathrm{Moon}}$$

The solution of these equations gives the following values, at mean moon distance:

$$\rho_{\mathrm{sdw}} = 0;43,34^\circ$$
$$\rho_{\mathrm{Moon}} = 0;16,46^\circ$$

Thus, the proportion between both values is 2;35,54 (2.5984).[44]

It is important to notice that the radius obtained for the Moon differs by only 6 seconds from the one that Ptolemy uses (0;16,40°), and the proportion between both is practically 2;36 (the difference is 0.0016). Actually, it is nearer to the proportion 2;36 than those same proportions calculated using the two pairs of eclipses, which Ptolemy had considered 'negligibly' different from 2;36. With these results, it is not absurd to think that the calculations performed with more eclipses would convince Ptolemy of the values he will finally adopt for the mean distance.[45]

However, this conjecture does not explain why the values that Ptolemy obtains are inconsistent with the distances. That is, if we keep in mind that the values at the mean distance (0; 16, 40° and 0; 43, 20°), correspond to 59^{tr}, the angles obtained at the eclipses for the apparent radius correspond to a maximum distance of 62; 45, 58^{tr} and a minimum of 55; 39, 37^{tr}. Obviously, the maximum and minimum distances obtained are nearer to the mean distance than the corresponding ones according to the system (64; 10^{tr} and 53; 50^{tr}). Why didn't Ptolemy calculate the distances correctly? Why did he stop at those distances?

The answer is: because at that distance, and only at that distance, would D_S be 1210^{tr}.

4.4 The inverted calculation

If Ptolemy knew that D_S must have a value of 1210^{tr}, the only thing that he could do was to follow the same procedure used in the calculation that was supposed to yield

---

44 The exact value of ρ_{Lun} is 0;16,46,30, and so, if Ptolemy had rounded it at 0;16,47, the value of the proportion would have been 2;35,45; but if he had maintained the exact value, it would have been 2;35,50.

45 Britton (1992) has proven that Ptolemy managed much more data than what indeed appears in the *Almagest*. If we had done the calculation with eclipse n°7($AP_7 = 0;55,5^\circ$), the results would have been $\rho_{\mathrm{Lun}} = 0;17,1^\circ$, $\rho_{\mathrm{som}} = 0;43,43^\circ$ and $\rho_{\mathrm{som}}/\rho_{\mathrm{Lun}} = 2;34,12$, and if we had done it with the average of both ($AP_{7,16} = 0;54,54^\circ$), the results would have been $\rho_{\mathrm{Lun}} = 0;16,53^\circ$, $\rho_{\mathrm{som}} = 0;43,38^\circ$ and $\rho_{\mathrm{som}}/\rho_{\mathrm{Lun}} = 2;35,3$. In this case, the calculation has been done taking the difference between the mean and the true anomalies into account, because at mean distance, the difference is significant. I note these results so that I am not accused of what we want to accuse Ptolemy; that is, of having chosen the data. Anyway, a) these results are also interestingly near to what I am looking for, b) there are reasons to prefer the data of eclipse n° 16—it cannot be excluded that Ptolemy has observed himself it but, even if he has not, for reasons of temporality eclipse n° 16 is much more reliable than the one which took place approximately 600 years before Ptolemy, and c) we are not saying that Ptolemy has obtained the value only from this calculation, but from several further calculations as well.

this value, but the other way round. Instead of introducing among the data, the value of the Moon's apparent radius (0;15;40°), he had to introduce the value of the Sun's distance (1210^{tr}). He would thus know the value for the Moon's apparent radius.

Actually, Ptolemy knows that (Fig. 1):

1. ND = 1210
2. ΘN = 64;10
3. NL = NM = 1
4. NM/HS = NG/HG = ND/ΘD
5. ND = NΘ + ΘD
6. ΘH = ΘS − HS
7. PR + ΘS = 2
   Now, from 5, 1 and 2, he obtains that
8. ΘD = ND − NΘ = 1210 − 64;10 = 1145;50
   And from 4, 1 and 8 that, as ND/ΘD = NM/HS, then: 1210/1145;50 = 1/HS, and so:
9. HS = 0;56,49°
10. From 6, 7 and 9 he knows that ΘH = 2 − PR − HS and that, therefore,
11. ΘH + PR = 1;3,11°.

Ptolemy could calculate, therefore, that the sum of the Moon's absolute radii and of the shadow's should be 1;3,11° in order to obtain 1210^{tr} for the Sun's distance.

Now, if we also assume that the proportion between both values is exactly 2;36, the only possible results are 0;17,33^{tr} and 0;45,38^{tr}, which correspond to the apparent radii of 0;15,40° and 0;40,44°, respectively. The value of the Earth's shadow does not coincide with the one established by Ptolemy, but this is because, in fact, as we have already said, he performed the calculation with the value of the proportion, not with 0;40,40°; the shadow's apparent radius does not play any role since when calculating D_S, only the proportion between the apparent radii ($\rho_{\mathrm{sdw}}/\rho_{\mathrm{Moon}}$) is used. Then, Ptolemy could have altered the shadow's value by removing a few seconds from it (exactly 4), in order to make the proportion to be 'negligibly less', as it should be at this distance. This would not imply any alteration in the Sun's distance, because in the calculation he will use the proportion and, therefore, 0;40,44°.

Once he obtained these values, Ptolemy could not calculate from them the values at the minimum distance because he knew that, indeed, these values did not correspond exactly to the maximum distances (64;10^{tr}), but to 62;45,58^{tr}. Since the Moon's apparent radius at the maximum distances is 0;15,40° and at the mean one is 0;16,40°, the radius at the minimum will obviously be 0;17,40°. In turn, since at the maximum distance the shadow's radius is 0;40,44° and at the mean one is 0;43,20°, so at the minimum, the radius will be 0;45,56°. However, since the proportion should now be 'negligibly greater', he will add to the shadow's value the same 4 seconds which he had subtracted to the value at the maximum distance and he will obtain 0;46°. A sign that the value Ptolemy had in mind was 0;45,56°, or 0;17,40° and the proportion was 2;36, is the following: one page after obtaining the values from the second pair of eclipses, when he sets out to obtain the limits of the lunar eclipses, instead of taking 0;46° as the radius of the shadow at the minimum distance, which he had just calculated, he multiplies 0;17,40° by the proportion 2;36 and obtains 0;45,56° (VI,5; H484; 286–287).

The only thing left for him was to find the eclipses which fitted with those data and, if he did not have them, he could force the calculations a little bit, as he presumably did in eclipse n° 4 when he transformed the 0;48,36 into 0;48,30, or make similar alterations in the second pair of eclipses.

There is one further detail. It is difficult to explain why Ptolemy affirms that the Moon's and the Sun's apparent radii coincide at the Moon's maximum distance, when the previous astronomers had affirmed that it was at the mean distance, denying the possibility of annular eclipses and forcing him, for example, to assert that the Moon at its minimum distance has a considerably bigger apparent radius than the Sun.[46] However, if my hypothesis is correct, there could be an explanation. If Ptolemy had calculated the distance of the Sun with the values $\rho_{\mathrm{sdw}} = 0;43,20°$, $\rho_{\mathrm{Moon}} = 0;16,40°$ and $D_L = 59^{\mathrm{tr}}$, the value of D_S would be 1919^{tr}, which Ptolemy could not accept. Clearly, he should have perceived, as he explicitly says in the *Hypotheses*, that "when we increase the distance to the Moon, we are forced to decrease the distance to the Sun, and *vice versa*. Thus, if we increase the distance to the Moon slightly, the distance to the Sun will be somewhat diminished and it will then correspond to the greatest distance of Venus" (Goldstein 1969: 7). Therefore, he took the equivalence of the radii at the Moon's maximum distances, so that the Sun would come closer.

5 Conclusion

Let us summarize the road that Ptolemy may have followed. He begins calculating the Earth–Sun distance making use of several eclipses. He obtains values which are absolutely incompatible with each other (due to the method's sensitivity) or, at least, values which are not compatible with what he expected, and so he decides to sort out the data. In doing so he needs to calculate the Earth–Sun distance with some other method, which he does by taking the planets apogee-perigee proportions and the Moon's distance. Rounding the values, Ptolemy obtains a minimum distance of 1160^{tr} and a mean distance of 1210^{tr}. From the calculation of many eclipses, he would have found that the proportion between the apparent radii of the shadow and of the Moon is always around 2;36, and so Ptolemy considers it as an extremely reliable value. He would have also obtained—perhaps from Hipparchus—a value for the Moon's radius at its mean distance: 0;16,40°. With those data he calculates the Sun's distance and he realizes that it is much bigger than 1210^{tr}, and he thus decides that the calculation should be made at the Moon's maximum distance and that, therefore, the coincidence between the Sun's and Moon's apparent radii will take place when the last one reaches its maximum distance and not its mean. With this value, assuming

[46] Cfr. Newton 1977: 192; Toomer [1984] 1998: 252, note 53; Neugebauer 1975: 104; Pedersen 1974: 208, mainly note 4.

that the Sun is at 1210^{tr}, and taking the proportion to be 2;36, Ptolemy finds the values 0;15,40° and 0;40, 40°, and, by means of them, he calculates the values that will correspond to the minimum distances: 0;17,40° and 0;46°. Then he chooses the eclipses that give these values and, if he does not find them exact, he modifies the values slightly. The fact that the eclipses are observed so far in the past raises the probability of the hypothesis.[47] Finally, when he writes the *Hypotheses*, taking into account that Mercury's proportions had already been modified, he knows that the result would not be consistent with the value of the *Almagest*. He was aware that Venus' maximum distance should be smaller than the Sun's minimum distance, but he does not realize that he makes a mistake in Mercury's proportion, because the result was close to what he expected. Ptolemy does not worry about rounding because he knows that the calculation has been made rounding the numbers. Finally, when he calculates Venus' absolute radius and orders the planets according to their volume, he makes another mistake and uses the old values of the previous calculation.

My hypothesis is able to explain several until now unanswered questions, some of which were discovered in this investigation: (1) why Ptolemy rounds the proportions in the calculation of the *Hypotheses*; (2) why Ptolemy does not realize the error he makes rounding the numbers when he calculates the Sun's distance in the *Hypotheses*; (3) why Ptolemy claims that the Sun's and the Moon's apparent radii coincide when the Moon reaches its maximum distance and not at its mean distance; (4) the apparent error in Venus' absolute radius and in the order of the volume of the planets; (5) the medieval correction of Venus' volume; (6) that the calculation with the correctly rounded numbers is exactly 1160; (7) that the Earth's shadow and Moon's proportion is exactly 2;36 at the mean distance, that it is 'negligibly less' at its maximum distance, and that it is 'negligibly larger' at its minimum distance; (8) that the Earth's shadow and the Moon's apparent radii are not consistent with the distances at which they are supposed to be; (9) why Ptolemy uses new eclipses to calculate the Earth–Sun distance when he already possessed two that perfectly fulfilled the requirements; (10) why the eclipses used to calculate the distance of the Sun are so far in the past.

Acknowledgements I would like to thank Dennis Duke, Bernard Goldstein, Alexander Jones, Noel Swerdlow and Albert van Helden for reading drafts of this article and sending me helpful comments and objections.

[47] Ptolemy usually uses very old data, but in general in one of these two situations: either he needs them to be far away in time in order to compare them with data he has obtained, since the longer time elapsed between the data he is using, the more precise the calculation he is carrying out will be (when he calculates, for example, the mean period of the planets); or he uses old data because they come from other authors, but in general, in this second situation, Ptolemy corroborates them with his own measurements (as when he calculates the Moon's epicycle). The present case is not comprised under any of these two possibilities.

Appendix: Chart of lunar eclipses

	N	Date				Toomer's page	magnitude	Time from T = 0				Earth/moon Distance		
		year	month	day	place			years	days	Hrs.	min.	Moon ubic.	Approximation's quality	
	1	-720	March	19	Bab.	191	Total	26	28	20	40	MEAN	Good	22.86
	2	-719	March	8	Bab.	191	3 S	27	17	23	10	MAX.	Good	12.40
	3	-719	Sept.	1	Bab.	192	> 6 n	27	194	19	40	MIN.	Good	16.99
Calculation of moon and shadow radii at maximum distance	4	-620	April	21	Bab.	253	3 S	126	86	16	45	MAX.	Good	19.883
	5	-522	July	16	Bab.	253	6 N	224	196	9	50	MAX.	Bad	27.90
	6	-501	Novem.	19	Bab.	208	3 N	245	327	10	15	MAX.	Very good	2.738
	7	-490	April	25	Bab.	206-7	2 S	256	122	10	15	MEAN	Good	10.32
	8	-382	Decem.	23	Bab.	211-2	Small	365	26	18	15	MEAN	Bad	42.28
	9	-381	June	18	Bab.	212	Parcial	365	203	7	50	MAX.	Bad	27.61
	10	-381	Decem.	12	Bab.	213	Total	366	15	9	50	MIN.	Very good	0.11
	11	-200	Sept.	22	Alex.	214	Without data	546	345	6	30	MEAN	Bad	30.20
	12	-199	March	19	Alex.	214	Total	547	158	13	20	MEAN	Good	20.58
	13	-199	Sept.	12	Alex.	215	Total	547	334	13	45	MEAN	Good	21.92
Calculation of moon and shadow radii at minimum distance	14	-173	May	1	Alex.	283	7 N	573	207	14	20	MIN.	Good	16.15
	15	-140	January	27	Rhodes	284	3 S	606	121	10	10	MIN.	Very good	1.22
	16	125	April	5	Alex.	206	2 S	871	256	8	5	MEAN	Good	18.12
	17	133	May	6	Alex.	198	Total	879	289	23	15	MAX.	Bad	40.05
	18	134	Oct.	20	Alex.	198	10 S	881	91	23	0	MEAN	Bad	25.02
	19	136	March	6	Alex.	198	6 N	882	229	4	0	MIN.	Bad	33.72

References

Al-Farghani, *elementa astronomica (liber de aggregatione scientiae stellarum)*. Ed. R. Campani, Collezione di opusculi danteschi, volumi 87–90 (1910).

Aristarchus. *Aristarchus of Samos on the Sixes and Distances of the Suna and Moon*. Translated by T. L. Heath ([1913] 1997): 351–411.

Britton, J. P. (1992) *Models and Precision: The Quality of Ptolemy's Observations and Parameters*. New York: Garland.

Carmody, F. J. (1960) *The Astronomical Works of Thabit B. Qurra*. California: The University of California Press.

Dreyer, J. L. (1917) "On the Origin of Ptolemy's Catalogue of Stars". *Monthly Notices of the Royal Astronomical Society* 77: 528–539.

Dreyer, J. L. (1918) "On the Origin of Ptolemy's Catalogue of Stars. Second Paper". *Monthly Notices of the Royal Astronomical Society* 78: 343–349.

Dreyer, J. L. (1953) *A History of Astronomy from Thales to Kepler*. Second edition, originally published as *History of the Planetary Systems from Thales to Kepler*. 1905. New York: Dover.

Evans, J. (1987) "On the origin of the Ptolemaic star catalogue", *J. Hist. Astronom.* 18: 155–172; 233–278.

Evans, J. (1998) *The History and Practice of Ancient Astronomy*. Oxford: Oxford University Press.

Festugière, A. J. (1968) *Proclus, Commentaire sur le Timée*. Tome Quatrième – Livre IV. Paris: Libraire Philosophique J. Vrin.

Gingerich, O. (1980) "Was Ptolemy a fraud?", *Quarterly Journal of the Royal Astronomical Society*. 21 (1980), 253–266.

Gingerich, O. (1981) "Ptolemy revisited: A reply to R R Newton", *Quarterly Journal of the Royal Astronomical Society*. 22: 40–44.

Goldstein, B. R. (1967) *The Arabic version of Ptolemy's Planetary Hypotheses*. Transactions of the American Philosophical Society, New Series, Vol. 57, part. 4

Grasshoff, G. (1990) *The history of Ptolemy's star catalogue*. New York: Springer.

Hartner, W. (1964) "Medieval views on Cosmic Dimensions and Ptolemy's Kitab al.Manshurat" in Koyré (1964), I: 254–282.

Hartner, W. (1980) "Ptolemy and Ibn Yûnus on Solar Parallax." *Archives Internationales d'Histoire des Sciences* 30: 5–26.

Heath, Th. ([1913] 1997) *Aristarchus of Samos. The Ancient Copernicus. A History of Greek Astronomy to Aristarchus together with Aristarchus' Treatise on the Sizes and Distances of the Sun and Moon*. (First edition: Oxford: Oxford University Press) Oxford: Oxford and Clarendon University Press.

Heiberg, J. L. (ed.) (1898–1903) *Claudii Ptolemaei Opera quae exstant omnia*. Vol. I, Syntaxis Mathematica, 2 vols. Leipzig: Teubner.

Heiberg, J. L. (ed.) (1907) *Claudii Ptolemaei Opera quae exstant omnia*. Vol. II, Opera Astronomica Minora. Leipzig: Teubner.

Ibn Rustah, Kitab al-Aʿlaq an-Nafisa VII, ed. M. J. de Goeje, *Bibliotheca Geographorum Arabicorum*, VII, Leiden (1982).

Jones, A. (2005) "Ptolemy's *Canobic Inscription* and Heliodorus' Observation Reports" *SCIAMVS* 6: 53–97.

Kepler, J. (1937-) *Johannes Kepler Gesammelte Werke*. Munich: C. H. Beck.

Koyré, A. (1964) Mélanges Alexandre Koyré, publiés à l'occasion de son soixante-dixième anniversaire, 2 vols. Vol I: *L'aventure de la science*. Paris: Hermann.

Lindberg, D. (1992) *The Beginnings of Western Science*. Chicago: The University of Chicago Press.

Nallino, C. A. (1899–1907) *Al-Battani sive Albatenii Opus Astronomicum* (Pubblicazione del reale osservatorio di Brera in Milano, 40, 3. vols.

Neugebauer, O. ([1957] 1969) *The Exact Sciences in Antiquity*. Second edition. New York: Dover.

Neugebauer, O. (1975) *A History of Ancient Mathematical Astronomy*. Studies in the History of Mathematics and Physical Sciences 1. 3 vols. Berlin: Springer.

Newton R. (1979) "On the fractions of degrees in an ancient star catalogue" *Quarterly Journal of the Royal Astronomical Society*. 20: 383–394.

Newton, R. (1973) "The authenticity of Ptolemy's parallax data, Part I" *Quarterly Journal of the Royal Astronomical Society* 14: 367–388.

Newton, R. (1974a) "The authenticity of Ptolemy's parallax data, Part II" *Quatrerly Journal of the Royal Astronomical Society* 15: 7–27.

Newton, R. (1974b) "The authenticity of Ptolemy's eclipse and star data" *Quarterly Journal of the Royal Astronomical Society* 15: 107–121.

Newton, R. (1977) *The Crime of Claudius Ptolemy*. Baltimore and London: John Hopkins University Press.

Pappus, *comentario al Almagesto*: see Rome (1931).

Pedersen, O. (1974) *A Survey of the Almagest*. Acta Historica Scientirarum Naturalium et medicinalium. Vol. 30. Odense: Odense University Press.

Pérez Sedeño, E. (1987) *Las Hipótesis de los Planetas*. Introducción y notas de E. Pérez Sedeño. Traducciones de J. G. Blanco y A. Cano Ledesma. Madrid: Alianza.

Proclus (1909) *Hypotyposis astronomicarum positionum*. Ed. Manitius. Leipzig: Teubner.
Ptolemy, C. *Almagest*. See Toomer (1998) and Heiberg (1898–1903)
Ptolemy, C. *The Planetary Hipotheses*. See Goldstein (1969), Heiberg (1907) and Pérez Sedeño (1987).
Ptolemy, C. *Canobic Inscription*. See Heiberg (1907) and Jones (2005).
Ptolemy, C. *Geography*. Claudii Ptolemaei Geographia, ed. C.F.A. Nobbe, 3 vols. Lepzig (1843–1845), reprinted in 1966.
Rome, A. (1931) *Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste*, 3 vols. (Tome I: Pappus d'Alexandrie: Commentaire sur les livres 5 et 6 de l'Almageste). Studi e Testi 54. Roma: Biblioteca Apostólica Vaticana.
Steele, J. M. (2000) "A Re-analysis of the Eclipse Observations in Ptolemy's Almagest" *Centaurus* 42: 89–108.
Swerdlow, N. M. (1992) "The enigma of Ptolemy's catalogue of stars", *J. Hist. Astronom.* 23 (3): 173–183.
Swerdlow, N. M. (1969) 'Hipparchus on the Distance of the Sun'. *Centaurus* 14: 287–305.
Swerdlow, N. M. (1968) *Ptolemy's Theory of the Distances and Sizes of the Planets: A Study in the Scientific Foundation of Medieval Cosmology*. Tesis Doctoral, Yale University.
Swerdlow, N. M. (1979) 'Ptolemy on Trial' *American Scholar* 48: 523–531.
Taub, L. C. (1993) *Ptolemy's Universe. The Natural Philosophical and Ethical Doundations of Ptolemy's Astronomy*. Chicago and LaSalle: Open Court.
Toomer, G. J. (1998) *Ptolemy's Almagest*. (First Edition: London: Durkworth, 1984) Princeton: Princeton University Press.
Toomer, G. J. (1974) "Hipparchus on the distances of the sun and moon", *Archives of History of Exact Sciences* 14: 126–142.
van Helden, A. (1986) *Measuring the Universe. Cosmic Dimensions form Aristarchus to Halley*. London: The University of Chicago Press.