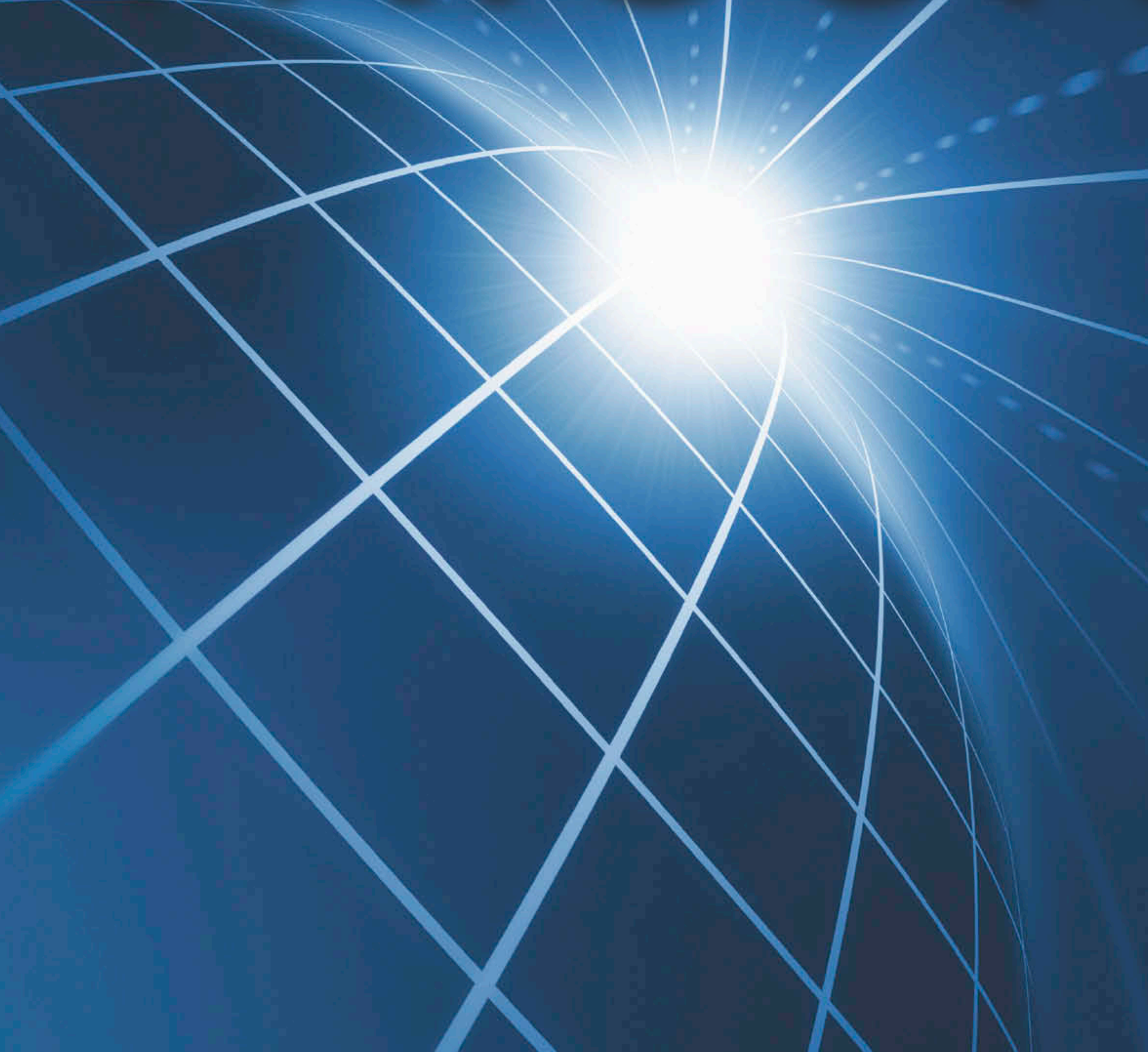
HISTORICAL REFLEC

TRIGON



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TIONS ON TEACHING OMETRY

Returning to the beginnings of trigonometry—the circle—has implications for how we teach it.

David M. Bressoud

The study of trigonometry suffers from a basic dichotomy that presents a serious obstacle to many students. On the one hand, we have triangle trigonometry, in which angles are commonly measured in degrees and trigonometric functions are defined as ratios of sides of a right-angled triangle. On the other hand, we have circle trigonometry, in which angles are commonly measured in radians and trigonometric functions are expressed in terms of the coordinates of a point on the unit circle centered at the origin. Faced with two such distinct conceptual approaches to trigonometry, is it any wonder that so many of our students get confused?

Once students begin to use the sine and cosine as examples of periodic functions, circle trigonometry dominates, but there is a tradition that triangle trigonometry is the simpler and more basic form and that students need to be grounded in this before being introduced to circle trigonometry. In fact, the historical evidence points in exactly the opposite direction.

Trigonometry—circle trigonometry—arose from the study of the heavens by the classical Greeks. It took more than a thousand years from the initial development of trigonometry by Hipparchus in the early second century BCE before triangle trigonometry was developed in earnest. An emphasis on triangles rather than circles is implicit in Al-Khwarizmi's work on shadows in the early ninth century CE, but this was not fully developed until Al-Biruni's *Exhaustive Treatise on Shadows* of 1021. Even so, applications of triangle trigonometry and, especially, the interpretation of trigonometric functions as ratios of sides of right-angled triangles did not achieve prominence until the sixteenth century. The switch in instructional emphasis from circle trigonometry to triangle trigonometry did not occur until the mid- to late-nineteenth century.

This historical overview of the development of trigonometry will present an argument for beginning the study of trigonometry with the circle definitions of the trigonometric functions and angle measurement. The historical information is based on Van Brummelen (2009) with additional details from Katz (2009) and Heath (1981); additional information on the history of trigonometry and classroom ideas can be found in Baumgart (1989).

THE FUNDAMENTAL PROBLEM OF TRIGONOMETRY

The fundamental problem from which trigonometry emerged is one that students seldom if ever see:

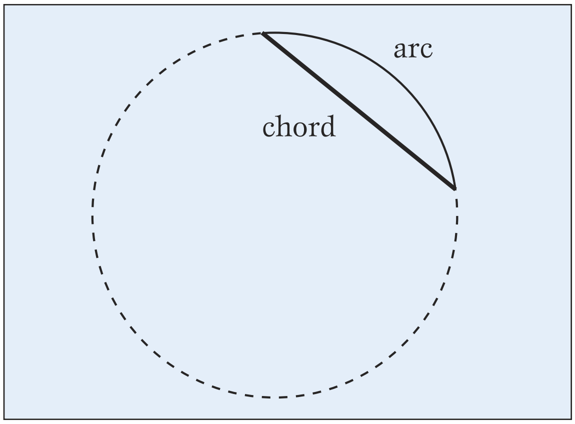

Fig. 1 Students in trigonometry are rarely asked to find the length of a chord intercepting an arc of a given length.

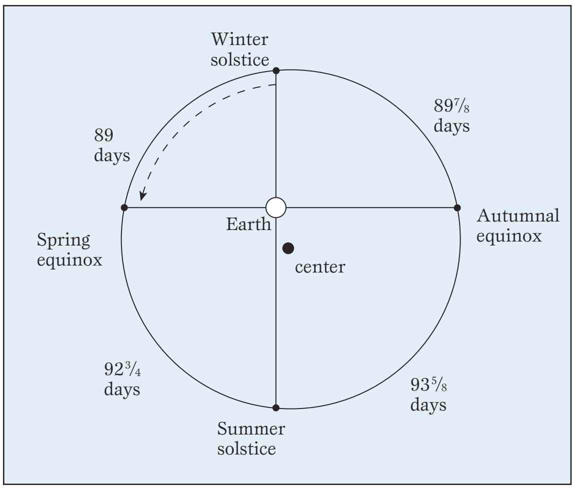

Fig. 2 Hipparchus used chord lengths to determine the distance from the earth to the center of the sun's orbit.

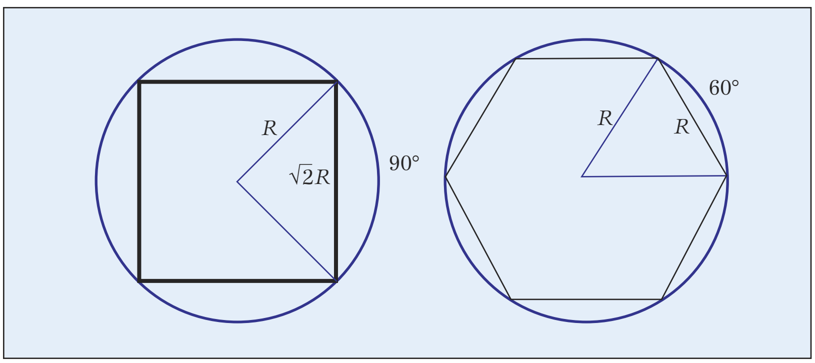

Fig. 3 The lengths of some chords are easily determined.

Given an arc of a circle, find the length of the chord that connects the endpoints of the arc (see **fig. 1**).

Perhaps the first problem that used what today we understand as trigonometry was one solved by Hipparchus of Rhodes (ca. 190–120 BCE): The seasons are of unequal length. Winter, at eighty-nine days, is the shortest; summer, at ninety-three and five-eighths days, is the longest. Hipparchus explained why.

Hipparchus used the observed lengths of the seasons to determine the length of the arc traveled by the sun in its orbit during each season. He then found the lengths of the chords that connect the sun's position at the ends of the seasons. These lengths enabled him to determine how far the earth is from the center of the sun's orbit (see **fig. 2**). (For a full description of this problem and its solution, see Hipparchus.pdf at www.nctm.org/mt.)

Arc lengths were usually measured in degrees, 360° being the full circumference of the circle. The length of the chord (denoted as crd) would depend on the radius, denoted by R. Some chords are easy to find (see **fig. 3**):

$$\text{crd } 180° = 2R$$
$$\text{crd } 90° = \sqrt{2}\,R \approx 1.414R$$
$$\text{crd } 60° = R$$

Euclid showed how to calculate other chord lengths when he determined the lengths of the sides of regular inscribed pentagons and decagons (see **fig. 4**):

$$\text{crd } 72° = \sqrt{\frac{5-\sqrt{5}}{2}}\,R \approx 1.176R$$
$$\text{crd } 36° = \frac{\sqrt{5}-1}{2}\,R \approx 0.618R$$

(For Euclid's proofs, see Euclid.pdf at www.nctm.org/mt.)

The general problem for astronomical work requires determining the chord length for any arc length. This is almost certainly the first example in mathematics history of a functional relationship without an explicit formula for calculating the output value for each input value. We are told that Hipparchus constructed a table of approximate values. The earliest such table that still survives was built by Ptolemy of Alexandria (90–168 CE) in his great work on astronomy, the *Mathematical Treatise,* better known by the Latinized version of its Arabic name, the *Almagest* (literally, *The Great Book*). Ptolemy constructed a table of chord lengths for a circle of radius 60 in half-degree increments. Ptolemy's work includes full proofs and requires some very impressive use of Euclidean geometry.

(For a description of Ptolemy's work, including proofs, see Ptolemy.pdf. at www.nctm.org/mt.)

What chord lengths have to do with trigonometry may not be immediately clear. If we rotate the circle so that the chord is vertical and insert a few radial lines, we can see how to translate chord lengths into the sine function: If the chord subtends an arc length of 2θ and the radius of the circle is R, then half the chord length is $R\sin\theta$. The chord of arc length 2θ is $2R\sin\theta$ (see **fig. 5**), or $\text{crd}(2\theta) = 2R\sin\theta$. Ptolemy's table is equivalent to a table of sines in quarter-degree increments. His calculations were carried out to seven-digit accuracy.

The shift to the half-chord, or sine, was made by Indian astronomers in the third, fourth, or fifth century CE. They also provided the source for our word *sine*. The Sanskrit word for *chord* was *jya*, and what we would refer to as the sine they named the *ardha-jya*, meaning "half-chord." In time, as astronomers adopted the practice of working only with the half-chord, the prefix was dropped, and just *jya* or *jiva* came to refer to the half-chord, or sine. When Arab astronomers learned Indian trigonometry, they transliterated *jiva* as *jyba*, with a spelling equivalent to *jyb* because the vowel *a* was not written. But *jyba* is not an Arabic word, and the word *jayb* is also spelled with the same three letters, *jyb*. By the time Arab trigonometric texts were translated into Latin, the pronunciation had shifted. *Jayb* can mean a fold or a bay; so European astronomers translated this word into the Latin *sinus*, which encompasses that meaning. From *sinus*, we get the English *sine*. But the term's true meaning is "half-chord."

THE DEVELOPMENT OF ANGLE MEASUREMENT

Today we define degrees as a fraction of a complete revolution, a characterization that is neither precise nor particularly clear. Until the late nineteenth century, degrees were considered a measure of arc length: 1° equals 1/360th of the circumference of the circle. One could speak of degrees as the measure of an angle between two intersecting line segments, but one would measure the angle by centering a circle at the intersection and determining the length of the arc between these segments (see **fig. 6**).

The size of a degree—because it is a fixed fraction of the circumference—depends on the radius of the circle. So does the length of the chord. Those who created tables of sines would pick a radius convenient to their calculations. This value was known as the *sinus totus* and could be read from the value of $\sin 90°$. Ptolemy used a radius of 60 because his fractions were expressed in 60ths (minutes from *pars minuta prima* = first small part), 60ths of 60ths (seconds from *secunda pars minuta* = second small part), and 60ths of 60ths of 60ths. Georg Rheticus (1514–74), the first European to publish tables of all six trigonometric functions, chose a radius of 1,000,000,000,000,000 = 10^{15}. This approach may seem strange until one realizes that it enabled him to create a table with fifteen-digit accuracy without recourse to decimals (not yet in common use) or fractions.

Indian astronomers also were the first to recognize that—because the sine relates two lengths—using the same units for both is convenient. If the

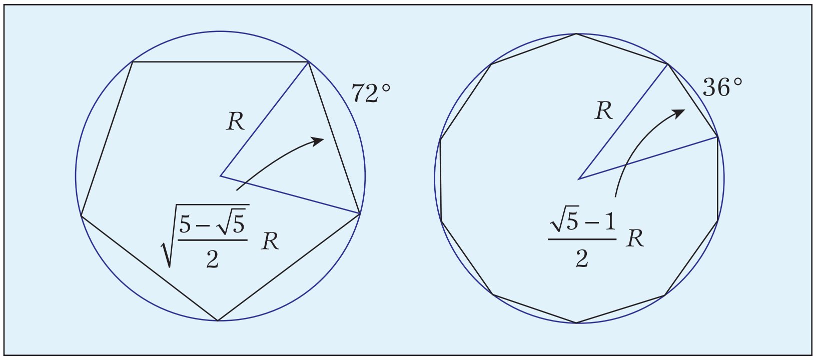

Fig. 4 Not until Euclid could geometers determine the side lengths of regular pentagons and decagons. Ptolemy used these measures to construct his table of chords. See the Web materials for how the golden ratio is involved.

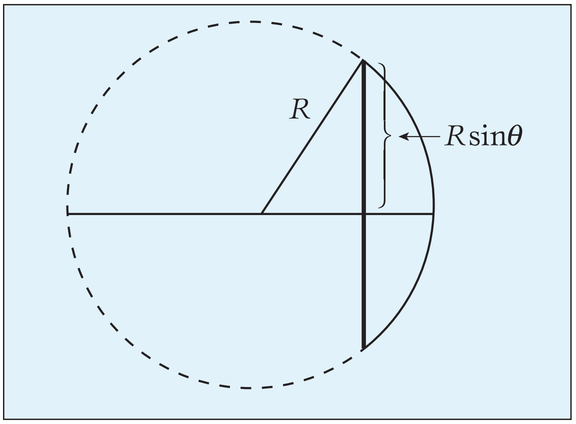

Fig. 5 When the chord intercepts an arc of 2θ, then its length is $2R\sin\theta$, where R is the radius of the circle.

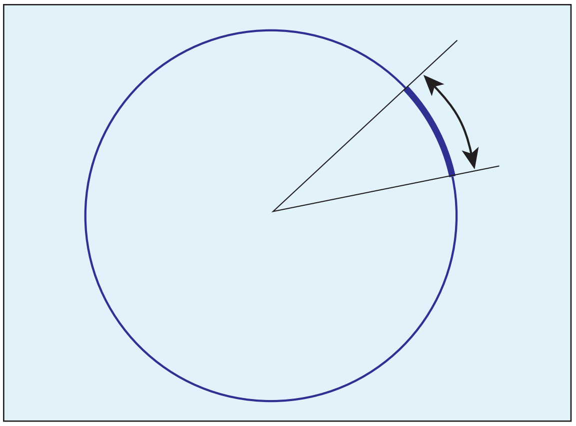

Fig. 6 Until the late 1800s, angles were measured by centering a circle at the intersection of two line segments.

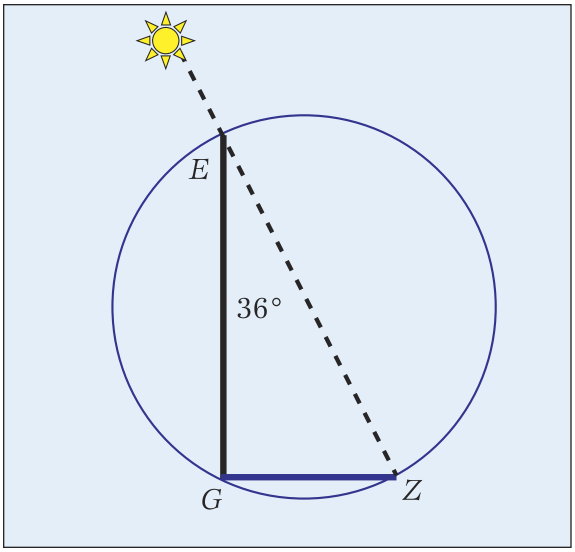

Fig. 7 Determining the length of a shadow cast by a vertical pole generated the study of triangle trigonometry.

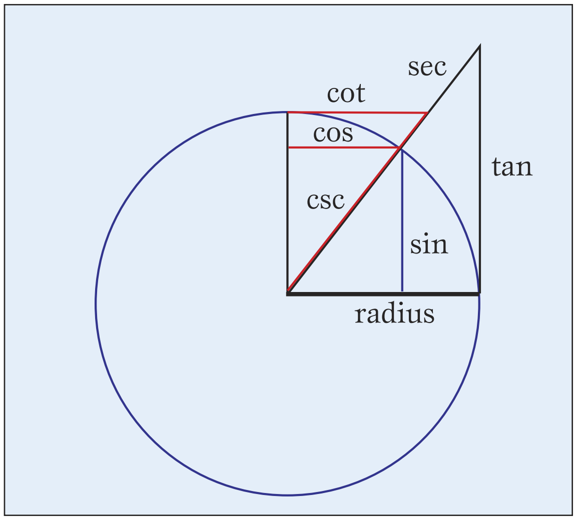

Fig. 8 All six trigonometric functions can be represented by line segments in the circle. The six functions did not appear jointly until the late tenth century.

circumference is 360°, then the radius should be $360/2\pi \approx 57.3°$. Decimals did not yet exist, and working with a mixed number is awkward, so Indian astronomers often chose instead to measure the circumference in minutes: $60 \cdot 360 = 21{,}600$. This approach gives a radius, or *sinus totus,* that could be taken as 3438. Measuring radius and arc length in the same units then enabled them to find polynomial approximations to the sine and cosine. For example, by the start of the fifteenth century it was known that for a radius of $R = 1$, the sine of θ could be approximated by $\theta - \theta^3/6$. In the following century, astronomers in Kerala in southwest India extended this result to find polynomial approximations of arbitrarily large degree as well as infinite series expansions for the sine and cosine.

It was Leonhard Euler (1707–83) who decided that the radius should be fixed at 1. He also realized that, for the purposes of calculus, measuring the arc and line lengths in the same units was absolutely essential. If the radius is 1, then the circumference is 2π. The arc length that had been described as 45° was now $\pi/4$. Euler might seem to be using radians, but he really was not. He was simply using the same units to measure the radius, the sine (or half-chord), and the arc that determined the sine. The term *radian* would not come into existence until the argument of the trigonometric functions had shifted from an arc length to an angle measured as a fraction of a turn—not until almost a hundred years after Euler's death.

THE EMERGENCE OF TRIANGLE TRIGONOMETRY

Triangle trigonometry began with the problem of determining the length of a shadow cast by a vertical stick, or *gnomon*, given the angle of the sun from vertical (see **fig. 7**). Ptolemy computed this at noon on the day of the equinox at latitude 36° (at which time the angle of inclination is 36°), but the earliest known table of such values was produced by Al-Khwarizmi of Baghdad (ca. 790–840). This is the same mathematician whose name is immortalized in our word *algorithm* and whose book on *al-jabr*, the source of our word *algebra*, is one of the most important milestones in the development of algebra.

Because the line length that we are calculating is tangent to the circle, this function came to be known as the *tangent*. The *secant*, arising from the Latin *secantem,* meaning "cutting," is the length of the radial line segment cut off by the tangent. The cosine, cotangent, and cosecant are the corresponding line segments for the complementary angle (see **fig. 8**). All six functions make their first joint appearance in the late tenth century in the commentary on the *Almagest* written by Abu'l Wafa (940–98), who also worked in Baghdad.

Applications of trigonometry to the calculation of sides of right triangles do not achieve prominence until 1533, with the posthumous publication of *De Triangulis Omnimodis* (*On Triangles of Every Kind*) by Johann Müller (1436–76), also known as Regiomontanus (see **fig. 9**). Bartholomew Pitiscus (1561–1613) is responsible for the word *trigonometry*; he chose *Trigonometria*, a Greek-to-Latin transliteration of "triangle measurement," for the title of his book. This book also marks the beginning of the common use of trigonometry in surveying. According to Katz, many of the trigonometry texts through the sixteenth century illustrated methods of solving plane triangles, "but not until the work of Bartholomew Pitiscus in 1595 did there appear any problem in such a text explicitly involving the solving of a real plane triangle on earth” (Katz 2009, p. 440).

Müller, Rheticus, and Pitiscus used trigonometry and similar triangles to solve for an unknown side of any given right triangle for which one of the acute angles and one other side are given. However, once Euler had fixed the radius of the defining circle at 1, it became possible to think of trigonometric functions as actual ratios of the sides. Eventually, it was discovered that if finding an unknown side of a right triangle is the main purpose for studying trigonometry, then the ratio definition is the most efficient means of defining these functions. The earliest textbook I have found that takes this approach was published in Germany in 1844 (Recht 1844). To my knowledge, the earliest such American textbook was published by an instructor at the U.S. Naval Academy in 1850 (Chauvenet 1855).

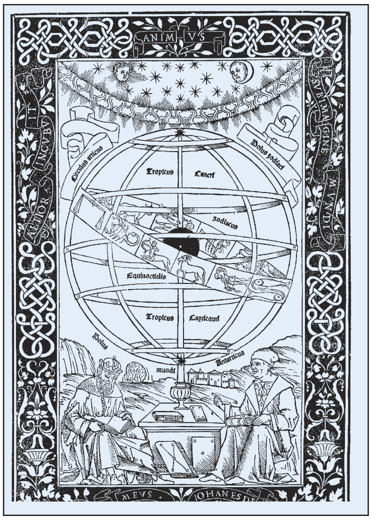

Fig. 9 *Epitome of the Almagest* (1496), by Johann Müller and George Peurbach, was their translation of and commentary on Ptolemy's *Almagest*. The frontispiece is shown.

The appearance of the unit we call *radian* provides evidence of when the ratio definition of trigonometric functions became pervasive. Once trigonometric functions are totally divorced from circles, it no longer makes sense to treat the argument as an arc length. Thus, a different approach to the measurement of an angle is needed, leading to the definition of a degree as a fraction of a complete revolution. The degree becomes 1/360th of a full turn, forcing practitioners to devise a name for the unit being used when 2π corresponds to a full turn. Although there is some uncertainty about who first coined the word *radian* as a contraction for “radial angle”—whether it was Thomas Muir or James Thomson, brother of Lord Kelvin—the term emerged sometime between 1869 (the earliest date claimed by Muir) and 1873 (the earliest appearance of this word in print by Thomson).

PEDAGOGICAL CONSIDERATIONS

In the mid-nineteenth century, when those studying trigonometry were most likely to use it in navigation and surveying, defining these functions as ratios made sense. There is convincing evidence that this approach does help students working on this type of problem (Kendal and Stacey 1998). But today students are more likely to encounter the sine and cosine as periodic functions rather than as navigational aids. Biological, physical, and social scientists use them more often to model periodic phenomena than to find the unknown side of a right triangle. If we want our students to understand trigonometric functions as functions, then the historical definitions that describe them as relating two lengths—arcs and line segments—are more transparent. Several researchers (Moore 2009; Thompson 2008; Weber 2005) have documented the confusion that arises when students and even teachers schooled in thinking of the sine as a numerical descriptor of a particular family of similar triangles try to stretch this understanding to one that encompasses a function of an angle.

It is no wonder that students have difficulty comprehending radians. One 360th of a “full turn” makes sense. We divide this revolution into 360 equal parts and take one of them. Few students can conceptualize what one 2πth of a full turn might be. Of course, $1/(2\pi)$ is a fraction that is mathematically meaningful, but it is also conceptually difficult. Much easier is to approach radian measure via Euler’s understanding of trigonometry, taking the argument of the sine as an arc length on a circle of radius 1 and then describing the value of the sine as the length of the corresponding half-chord.

History has much to teach us, and we ignore at our peril the historical route by which our ancestors were led to discover important mathematical ideas. We would do well to introduce trigonometry by imitating the astronomers who first discovered and explored these functional relationships by seeing them as connecting lengths of arcs and lengths of line segments. This is not to downplay the importance of triangle trigonometry or the understanding of the trigonometric functions as ratios, but if trigonometric functions are first introduced as lengths of line segments in a circle of radius 1, then they have a concrete meaning. From this, it is then possible to argue from similar triangles that these functions can also represent ratios. For students who first memorize trigonometric functions as ratios, making the transition to seeing them as lengths is much harder.

We can learn from Henri Poincaré, who advised, “The task of the educator is to make the child’s spirit pass again where its forefathers have gone, moving rapidly through certain stages but suppressing none of them. In this regard, the history of science must be our guide” (1899, p. 159).

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For the author’s full descriptions of the problems and solutions presented by Hipparchus, Euclid, and Ptolemy, go to the *Mathematics Teacher* Web site: www.nctm.org/mt.

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Supplement to "Historical Reflections on Teaching Trigonometry": Hipparchus

David M. Bressoud

March 13, 2010

We do not know the very first problem that used trigonometry in the sense of using the measure of an angle to find the length of a line segment. In the third century BCE, Aristarchus used angle measurements to estimate the distance to the moon and Archimedes estimated the width of the sun (see Van Brummelen, pp. 20–32). But neither of them used a table of values of chords or sines. Both used the fact that the angles they were considering were extremely small, and thus the chord length could be approximated by the arc length. Ptolemy credits the first table of chord lengths to Hipparchus and also credits him with the solution to the problem of the unequal seasons, perhaps the earliest problem that was solved using such a table.

An observation that was noted by Aristotle and that puzzled ancient astronomers is that the seasons are not of equal length. During the course of the year, the sun travels along the *ecliptic*, the circular path through the heavens that proceeds through the constellations that are recorded as the signs of the zodiac. The positions of the sun at the winter and summer solstices were observed to be diametrically opposite points on this circle. Moving out at right angles marked the spring and autumnal equinoxes, the half way points between the solstices. Together, the solstices and equinoxes were chosen to mark the changes of the seasons. Since the earth was assumed to be the center of the universe with the sun making its annual trajectory along this circular path, the seasons should be of equal length. They are not.

Modern calculations give the following approximate values to the lengths of the seasons:

winter 89 days,

spring $92\frac{3}{4}$ days,

summer $93\frac{5}{8}$ days,

fall $89\frac{7}{8}$ days.

Hipparchus of Rhodes (*circa* 190–120 BCE) explained this discrepancy by moving the earth off the center of the universe so that the perpendicular chords marking out the seasons do

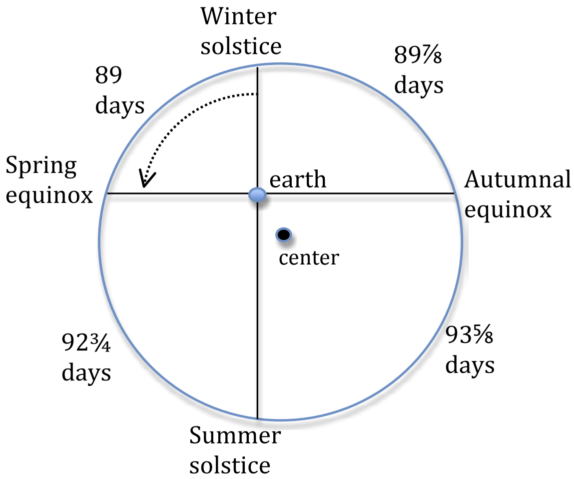

Figure 1: Hipparchus modeled the sun's path by moving the earth away from the center.

not cut arcs of equal length (see Figure 1). That raised the natural question: How far off center *is* the earth?

To answer this, we first convert the length of each season into the length of the corresponding arc, where the full circumference has length 360°. For example, the arc length of winter is $89/365\frac{1}{4}$ of the circumference. In terms of degrees, this is

$$\frac{89}{365.25} \cdot 360 \approx 87 + \frac{43}{60},$$

or approximately $87^\circ\ 43'$. In terms of arc length, the seasons are

winter $87^\circ\ 43'$,

spring $91^\circ\ 25'$,

summer $92^\circ\ 17'$,

fall $88^\circ\ 35'$.

Fall and winter together account for an arc length of $176^\circ\ 18'$, which means that the arc length from the spring equinox to the horizontal diameter of the sun's path is $1^\circ\ 51'$ (see Figure 2). If we can find the chord of $3^\circ\ 42'$, then half that chord is the vertical displacement of the earth from the center of the universe.

Of course, the actual distance depends on the radius of this circle. The radius is the average distance between the earth and the sun, which is 1 *astronomical unit* (au)—known

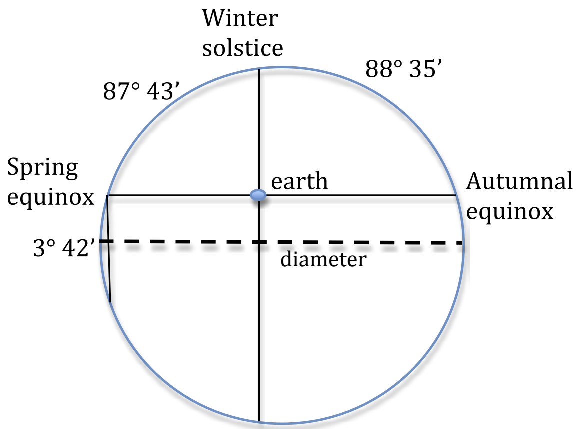

Figure 2: Arc lengths of the sun's path (not to scale).

today to be about 150 million km. Using the values of the sine function, we can calculate the chord length (crd) to be

$$\text{crd}\ (3^\circ\ 42') = 2\sin(1^\circ\ 51') = 2\sin 1.85^\circ = 0.065\ \text{au}.$$

The vertical displacement is half that, about 0.032 au.

Winter and spring account for 179° 08′. The arc length from the summer solstice to the vertical diameter of the sun's path is only 26 minutes. The horizontal displacement is approximately half the chord of 52′:

$$\text{crd}\ (52') = 2\sin(26') = 2\sin 0.433^\circ = 0.015\ \text{au},$$

for a horizontal displacement of 0.0075 au.

By the Pythagorean theorem, the distance from the earth to the center of the universe is about $\sqrt{0.0075^2 + 0.015^2} = 0.017$ au.

Exactly the same mathematics can be used if we assume that the earth circles the sun in a circular orbit at constant angular velocity. The sun would be about 2.55 million km from the center of the earth's orbit. But that is not what actually happens. The earth's orbit is elliptical, and the earth speeds up as it gets closer to the sun and slows down as it recedes. In fact, the sun, which is located at one of the foci of the earth's orbit, is just a little less, about 2.5 million km, from the center of the earth's orbit.

Supplement to "Historical Reflections on Teaching Trigonometry": Euclid

David M. Bressoud

June 9, 2010

When Ptolemy constructed his table of chords, he was able to start with the chords of arc lengths 180°, 90° and 60° as well as two chords that can be found in Euclid's *Elements*:

$$\text{crd } 36^\circ = \frac{\sqrt{5}-1}{2}\,R, \quad \text{and} \quad \text{crd } 72^\circ = \sqrt{\frac{5-\sqrt{5}}{2}}\,R,$$

where R is the radius of the circle. This is not quite the way that Euclid stated these results. These results are contained in Book XIII, Propositions 9 and 10:

Book XIII, Proposition 9. *If the side of the hexagon and that of the decagon inscribed in the same circle are added together, then the whole straight line has been cut in extreme and mean ratio, and its greater segment is the side of the hexagon.*

Book XIII, Proposition 10. *If an equilateral pentagon is inscribed in a circle, then the square on the side of the pentagon equals the sum of the squares on the sides of the hexagon and the decagon inscribed in the same circle.*

We first will see how to interpret these statements as chord lengths of the respective angles. We then will prove the propositions.

The side of the inscribed hexagon is the chord of 60°, which is R, the radius of the circle. The chord of the inscribed decagon is the chord of 36°. To say that a line segment has been cut in "mean and extreme proportion" means that the ratio of the longer to the shorter length is the golden ratio: $(1+\sqrt{5})/2$. Proposition 9 states that

$$\frac{R}{\text{crd } 36^\circ} = \frac{1+\sqrt{5}}{2}.$$

Equivalently,

$$\text{crd } 36^\circ = \frac{R}{(1+\sqrt{5})/2} = \frac{2R}{\sqrt{5}+1}\cdot\frac{\sqrt{5}-1}{\sqrt{5}-1} = \frac{2R(\sqrt{5}-1)}{5-1} = \frac{\sqrt{5}-1}{2}\,R.$$

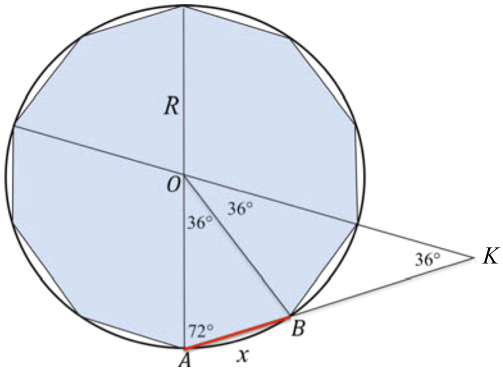

Figure 1: The ratio of the radius to the chord of 36°, R/x, is equal to $(1+\sqrt{5})/2$.

The side of the inscribed pentagon is the chord of 72°. Proposition 10 implies that

$$\begin{aligned}
(\text{crd } 72^\circ)^2 &= R^2 + \left(\frac{\sqrt{5}-1}{2}\,R\right)^2 \\
&= \left(1 + \frac{(\sqrt{5}-1)^2}{2^2}\right) R^2 \\
&= \left(1 + \frac{6-2\sqrt{5}}{4}\right) R^2 \\
&= \frac{10-2\sqrt{5}}{4}\,R^2 \\
&= \frac{5-\sqrt{5}}{2}\,R^2.
\end{aligned}$$

The value of crd 72° is found by taking the square root of each side.

The proofs of these propositions, while following those of Euclid, have been cast into modern terminology. A direct translation of Euclid's proofs can be found at http://aleph0.clarku.edu/∼djoyce/java/elements/toc.html

Proof of Proposition 9. See Figure 1. The length of $\overline{AB}$, one side of the inscribed decagon, is denoted by x. Since triangle OAB is isosceles, $\angle OAB = 72°$. Extend the line segment $\overline{AB}$ and the radial line to the next vertex of the decagon so that they meet at point K. The angle at K is $\angle AKO = 36°$.

Since triangle OBK is isosceles, $BK = R$. By the similarity of triangles OAK and

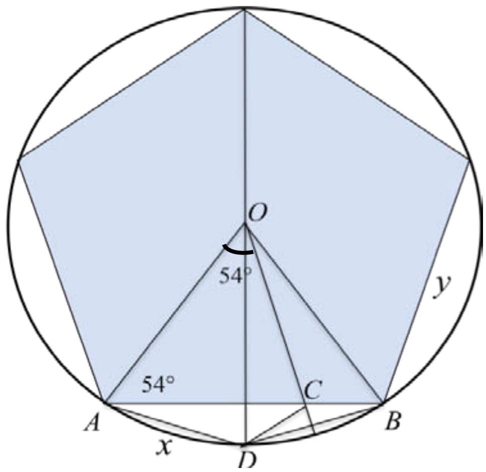

Figure 2: The square of the chord of 72° is equal to the sum of the squares of the radius and of the chord of 36°: $y^2 = R^2 + x^2$.

ABO, we obtain the relationship

$$\frac{R}{x} = \frac{R+x}{R} = 1 + \frac{x}{R}.$$

Let $z = R/x$. The equation is $z = 1 + z^{-1}$, which Euclid immediately recognized as the equation of mean and extreme proportion. In modern algebraic notation, we convert this to the quadratic equation $z^2 - z - 1 = 0$ which can be solved for z:

$$\frac{R}{x} = z = \frac{1+\sqrt{5}}{2}.$$

□

We now know that the chord of 36° is $R\left(\sqrt{5}-1\right)/2$.

Proof of Proposition 10. See Figure 2. The length of $\overline{AB}$, one side of the inscribed pentagon, is denoted by y. The length of $\overline{AD}$, one side of the inscribed decagon, is denoted by x. Draw the perpendicular bisector of $\overline{DB}$, and denote by C the point at which it intersects $\overline{AB}$. It follows that $\angle OAB = \angle AOC = 54°$, and therefore triangles OAB and COA are similar isosceles triangles. Therefore,

$$\frac{R}{y} = \frac{AC}{R} \quad \Longrightarrow \quad R^2 = y \cdot AC. \tag{1}$$

Triangles DAB and CDB are also similar isosceles triangles, and therefore

$$\frac{x}{BC} = \frac{y}{x} \quad \Longrightarrow \quad x^2 = y \cdot BC. \tag{2}$$

Combining equations (1) and (2) yields the desired result,

$$R^2 + x^2 = y\,(AC + BC) = y^2. \tag{3}$$

□

This establishes that the chord of 72° is $\sqrt{\dfrac{5-\sqrt{5}}{2}}\,R$.

Supplement to "Historical Reflections on Teaching Trigonometry": Ptolemy

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March 13, 2010

Claudius Ptolemy of Alexandria (*circa* 85–165 CE) established the basis for Western, South Asian, and Middle Eastern astronomy that would last until the 16th century. Ptolemy's book, known originally as "The Mathematical Treatise," would come to be known as "The Great Treatise" or, in Arabic, *Kitāb al-majisī*, which was translated into Latin as the *Almagest*. One of the basic challenges that Ptolemy had to face was constructing a table of chords corresponding to various arc lengths.

First he had to choose the radius of his circle, recognizing that chord lengths would need to be scaled when applied to specific circles. He chose a radius of $R = 60$. Since, following the practice of the ancient Mesopotamians, each degree was subdivided into 60 minutes and each minute into 60 seconds and so on, the choice of 60 simply makes it easy to scale, much as choosing a radius of 100 would be convenient for our decimal system. We will explain his results in terms of an arbitrary radius, R.

As explained in the print article and the supplement on Euclid, Ptolemy started with a knowledge of the chords (crd) of several arc lengths, especially crd $60° = R$ and crd $72° = \sqrt{(5-\sqrt{5})/2}\,R$. The next step in building his table was to show how to find the chord for an arc length that is the sum or difference of arc lengths for which the chords are known. This would enable him to find the chord length for $12°$, and then successively cut that in half to get down to the chord of $3/4^{\text{ths}}$ of a degree, not quite what he needed, but getting close. The key to the sum and difference of arc lengths formula is what we now know as Ptolemy's Theorem.

Ptolemy's Theorem. *Given any quadrilateral inscribed in a circle, the product of the diagonals equals the sum of the products of the opposite sides.*

Proof. The proof relies on Euclid's result, Book III, Proposition 21, that if we take any chord $\overline{AB}$ of a circle and any third point C on the circle, then the angle $\angle ACB$ depends only on the chord $\overline{AB}$ and not on the choice of C. In fact, $\angle ACB$ is exactly half the length of the arc from A to B, a result that we shall need later. It follows that in Figure 1,

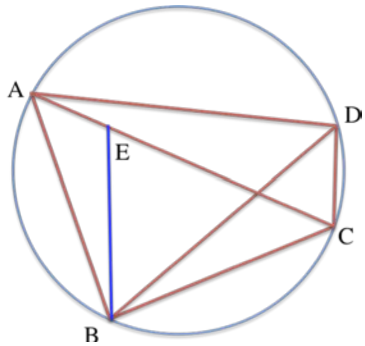

Figure 1: For any quadrilateral inscribed in a circle, the product of the diagonals is equal to the sum of the products of the opposite sides: $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

$\angle BAC = \angle BDC$. We draw a line segment from B, meeting $\overline{AC}$ at E, so that $\angle ABE = \angle DBC$. It follows that triangles ABE and DBC are similar, and, therefore

$$\frac{AE}{AB} = \frac{CD}{BD} \quad \Longrightarrow \quad BD \cdot AE = AB \cdot CD.$$

Again invoking Book III, Proposition 21, we see that $\angle ADB = \angle ACB$. From the construction of $\overline{BE}$, we also have that $\angle ABD = \angle CBE$. Now it follows that triangles ADB and ECB are similar, and, therefore

$$\frac{BD}{AD} = \frac{BC}{EC} \quad \Longrightarrow \quad BD \cdot EC = AD \cdot BC.$$

Combining these results, we obtain

$$BD \cdot AC = BD \cdot (AE + EC) = BD \cdot AE + BD \cdot EC = AB \cdot CD + AD \cdot BC. \qquad (1)$$

□

To get the sum and difference of angles formulas, we consider the special case of this theorem in which one of the diagonals of the quadrilateral is a diameter of the circle of radius R (see Figure 2 in which α is the arc length from A to D and β is the arc length from A to B. The diameter is $AC = 2R$). Note that for any arc length α,

$$(\text{crd } \alpha)^2 + \big(\text{crd } (180^\circ - \alpha)\big)^2 = (2R)^2.$$

If we know the lengths of chords $\overline{AB}$ and $\overline{AD}$, then we know the lengths of all chords except $\overline{BD}$, and Ptolemy's Theorem can be used to find this chord length (crd):

$$\text{crd } (\alpha) \cdot \text{crd } (180^\circ - \beta) + \text{crd } (\beta) \cdot \text{crd } (180^\circ - \alpha) = 2R \,\text{crd } (\alpha + \beta). \qquad (2)$$

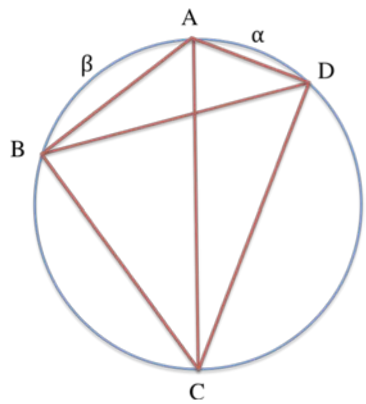

Figure 2: In terms of chords (crd), the sum of angles formula for the sine translates as crd $(\alpha) \cdot$ crd $(180° - \beta) +$ crd $(\beta) \cdot$ crd $(180° - \alpha) = 2R\,$crd $(\alpha + \beta)$, a direct consequence of Ptolemy's Theorem.

Similarly, if we know the chord of α and the chord of $\alpha + \beta$, Ptolemy's Theorem enables us to find the chord of β.

Ptolemy could have derived the half angle formula by setting $\alpha = \beta$ in equation (2) and solving for crd α in terms of crd 2α. He chose instead to derive the formula

$$(\text{crd } \alpha)^2 = 2R^2 - R\,\text{crd } (180° - 2\alpha) \tag{3}$$

as follows (see Figure 3). We draw a diameter $\overline{AB}$ and place points C and D so that the arc length from B to C and the arc length from C to D both equal α. We drop a perpendicular from C to $\overline{AB}$, meeting $\overline{AB}$ at F, and locate the point E so that $AD = AE$. Since $\angle DAC = \angle CAE$, triangles ADC and AEC are congruent. Since $CE = CD = BC$, triangles CFE and CFB are congruent, implying that

$$BF = \frac{1}{2}BE = \frac{1}{2}(AB - AE) = \frac{1}{2}(AB - AD). \tag{4}$$

Triangles ACB and CFB are similar, so

$$\frac{BC}{BF} = \frac{AB}{BC} \quad \Longrightarrow \quad BC^2 = AB \cdot BF = \frac{AB}{2}(AB - AD). \tag{5}$$

Using the fact that $AB = 2R$, $BC =$ crd α, and $AD =$ crd $(180° - 2\alpha)$, we get equation (3).

The results obtained so far enabled Ptolemy to find the exact value of the chord of any arc of length $3 \cdot 2^k$ degrees where k can be any integer. In the 12th century, this fact would lead Al-Samawal to argue that the circle should be divided into 480° rather than 360°,

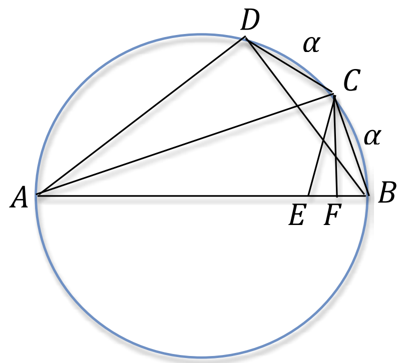

Figure 3: In terms of chords, the half angle formula can be expressed as $(\text{crd }\alpha)^2 = 2R^2 - R\,\text{crd }(180° - 2\alpha)$.

because it *is* possible to find an exact value for the chord of $1/480^{\text{th}}$ of the circumference of a circle. He seems to have convinced no one to change the definition of a degree.

Although Ptolemy could not find an exact value for the chord of $1°$, he was able to create a table of chord lengths for all arc lengths from $0°$ to $90°$ in increments of half a degree and to within an accuracy of one part in $216{,}000 = 60^3$. The following proposition enabled him to calculate crd $1°$ and crd $30'$ to the desired accuracy.

Proposition. *If* $0 < \alpha < \beta < 180°$ *are arc lengths, then*

$$\frac{\text{crd }\beta}{\text{crd }\alpha} < \frac{\beta}{\alpha}. \tag{6}$$

In particular, this implies that

$$\frac{2}{3}\,\text{crd }1°\,30' < \text{crd }1° < \frac{4}{3}\,\text{crd }45',$$

bounds that produce the desired accuracy. In fact, these bounds differ by less than one part in 2,600,000.

Proof. See Figures 4 and 5. Let α denote the arc length from A to B and β the arc length from B to C. Draw the line segment $\overline{BD}$ that bisects $\angle ABC$, and mark E as the point of intersection of $\overline{AC}$ and $\overline{BD}$.

We will need the fact that, since $\angle ABE = \angle EBC$,

$$\frac{AB}{AE} = \frac{BC}{CE}. \tag{7}$$

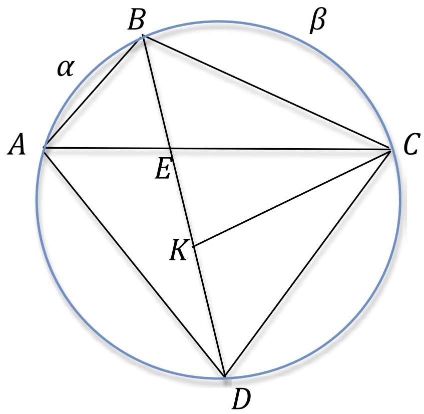

Figure 4: If ABC is any triangle and $\overline{BE}$ bisects the interior angle at B, then the ratio of AB to AE is equal to the ratio of BC to CE: $\dfrac{AB}{AE} = \dfrac{BC}{CE}$.

We can see why this is true if we add a point K on the segment $\overline{BD}$ so that $CE = CK$. Since triangle ECK is isosceles, $\angle EKC = \angle KEC = \angle AEB$, and therefore triangles ABE and CBK are similar. From this it follows that

$$\frac{AB}{AE} = \frac{BC}{CK} = \frac{BC}{CE}.$$

We draw a perpendicular from D to $\overline{AC}$, meeting $\overline{AC}$ at F (see Figure 5). Since the arc from A to D is equal to the arc from D to C, F is at the midpoint of $\overline{AC}$. We draw the arc of the circle centered at D with radius DE and mark its intersection with $\overline{AD}$ as G and its intersection with the extension of $\overline{DF}$ as H.

Since triangles EFD and AED have the same heights, the ratio of their areas is equal to the ratio of their bases, EF/AE. The ratio of the areas of these triangles is less than the ratio of the areas of the sectors EHD to GED, and the ratio of the sectors is equal to the ratio of the angles at D. We see that

$$\frac{EF}{AE} = \frac{\text{area of } \Delta EFD}{\text{area of } \Delta AED} < \frac{\text{area of sector } EHD}{\text{area of sector } GED} = \frac{\angle EDF}{\angle ADE}. \tag{8}$$

We observe that

$$EC = EF + FC = EF + AF = 2EF + AE,$$

and, similarly, $\angle EDC = 2\angle EDF + \angle ADE$. Now we put it all together,

$$\begin{aligned} \frac{\text{crd } \beta}{\text{crd } \alpha} &= \frac{BC}{AB} = \frac{EC}{AE} = 2\frac{EF}{AE} + \frac{AE}{AE} \qquad (9) \\ &< 2\frac{\angle EDF}{\angle ADE} + \frac{\angle ADE}{\angle ADE} = \frac{\angle EDC}{\angle ADE} = \frac{\beta/2}{\alpha/2} = \frac{\beta}{\alpha}. \end{aligned}$$

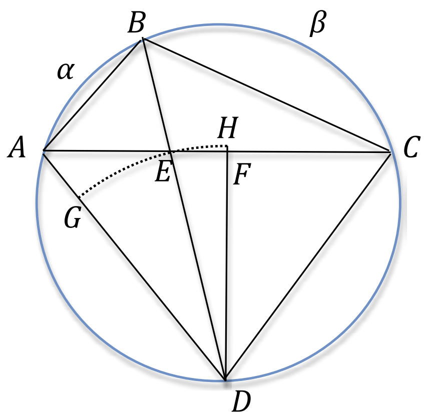

Figure 5: For ratios larger than 1, the ratio of the chord lengths is strictly less than the ratio of the corresponding arc lengths: $\dfrac{\text{crd } \beta}{\text{crd } \alpha} < \dfrac{\beta}{\alpha}$.

□

Now that Ptolemy knew the chord of $\frac{1}{2}^\circ = 30'$ and had exact values at each multiple of $1^\circ\,30'$, he could find very accurate values for chords at any multiple of half a degree. Of course, as we saw in the problem of locating the position of the earth, we need finer values than this. In his table, Ptolemy reported the value of one sixtieth of the difference between each pair of successive chord values. With this information, anyone using his table could employ linear interpolation to find the intermediate chord values for arc lengths in increments of half a minute.