### 9.2. Undoing the Chain Rule

This chapter focuses on finding accumulation functions in closed form when we know its rate of change function. We've seen that 1) an accumulation function in closed form is advantageous for quickly and easily generating many values of the accumulating quantity, and 2) the key to finding accumulation functions in closed form is the Fundamental Theorem of Calculus. The FTC says that an accumulation function is the antiderivative of the given rate of change function (because rate of change is the derivative of accumulation).

In Chapter 6, we developed many derivative rules for quickly finding closed form rate of change functions from closed form accumulation functions. We also made a connection between the form of a rate of change function and the form of the accumulation function from which it was derived.

Rather than inventing a whole new set of techniques for finding antiderivatives, our mindset as much as possible will be to use our derivatives rules in reverse to find antiderivatives. We worked hard to develop the derivative rules, so let's keep using them to find antiderivatives!

Thus, whenever you have a rate of change function $f$ and are charged with finding its antiderivative, you should frame the task with the question "What function has $f$ as its derivative?"

For simple rate of change functions, this is easy, as long as you know your derivative rules well enough to apply them in reverse. For example, given a rate of change function ....
$\ldots 2 x$, what function has $2 x$ as its derivative? i.e. if $f^{\prime}(x)=2 x$, then $f(x)=\boldsymbol{x}^{2}$ (plus any constant)
$\ldots-\sin x$, what function has $-\sin x$ as its derivative? $\cos \boldsymbol{x}+C$
$\ldots e^{x}$, what function has $e^{x}$ as its derivative? $\boldsymbol{e}^{x}+C$
Several more examples like these were given in 9.1 exercise 2? 3?. It would be a good idea to review those now for practice before continuing on.

## Unfortunately, Not All Antiderivatives Are Easy....

In that list from exercise $2 / 3$ were some slightly harder examples similar to this:

What's the principal antiderivative of the rate of change function $3 x^{2} e^{x^{3}}$ ? Again, to tackle this, ask the question "What function $f$, if I took the derivative of it, would yield $f^{\prime}(x)=3 x^{2} e^{x^{3}} ?$ "

Since $f^{\prime}$ is the product of two functions, you might conjecture that $f(x)=x^{3} e^{x^{3}}$, reasoning that the antiderivative of $f^{\prime}$ is the product of antiderivatives. But checking this answer by taking its derivative, which requires the product rule, yields a sum that is entirely different than the function $f^{\prime}$ given.

If, then .
The conjecture is incorrect.
Someone with a strong knowledge of the derivative rules can recognize that $3 x^{2} e^{x^{3}}$ is a derivative found by using the chain rule, since $3 x^{2}$ is the derivative of $x^{3}$, the argument of the composite function $e^{x^{3}}$. So, what function, if the chain rule is applied, yields $3 x^{2} e^{x^{3}}$ ? The answer is $e^{x^{3}}$. Finding the derivative of $e^{x^{3}}$ with the chain rule verifies that it is indeed the antiderivative of $3 x^{2} e^{x^{3}}$ 。

This section focuses on determining antiderivatives similar to this example. This way of thinking is developed below and is called Undoing the Chain Rule.

## Undoing the Chain Rule

The chain rule is for determining derivatives of composite functions; a composite function has the form $h(x)=g(f(x))$. Recall the chain rule is...
$r_{h}(x)=r_{g}(f(x)) r_{f}(x) \quad$ or $\quad h^{\prime}(x)=g^{\prime}(f(x)) f^{\prime}(x)$
We can summarize the process symbolically as shown below.

## CHAIN RULE:

## Derivative

$$
g(f(x)) \quad \rightarrow \quad g^{\prime}(f(x)) f^{\prime}(x)
$$

Thus, undoing the chain rule means first recognizing when a rate of change function has the structure $g^{\prime}(f(x)) f^{\prime}(x)$, and then determining the antiderivative by "recovering" the original function $g(f(x))$, as shown below.

## UNDOING THE CHAIN RULE:

## Anti-Derivative

$$
g^{\prime}(f(x)) f^{\prime}(x) \quad \rightarrow \quad g(f(x))
$$

## The Role of Constant Factors \& Coefficients

Here is the idea and pattern of Undoing the Chain Rule applied to the example above:


This works because $3 x^{2}$ is the derivative of $x^{3}$, according to the result of applying the chain rule to $e^{x^{3}}$.

Notice that functions like $5 x^{4} e^{x^{3}}, x e^{x^{3}}$, and $x^{-2} e^{x^{3}}$ are not eligible for undoing the chain rule, because applying the chain rule to $e^{x^{3}}$ results in the factor $3 x^{2}$, which necessarily has power 2 .

Now consider $x^{2} e^{x^{3}}$ or $10 x^{2} e^{x^{3}}$. Are these eligible for undoing the chain rule, although they don't have a coefficient of 3 ?

Yes -- once the requirement of power 2 is met, we can determine the coefficient for the answer (i.e. the antiderivative) so that its derivative is the same as the given rate function.

Specifically, if the antiderivative of $3 x^{2} e^{x^{3}}$ is $e^{x^{3}}$, then the antiderivative of $x^{2} e^{x^{3}}$ (which is $1 / 3$ as much as $3 x^{2} e^{x^{3}}$ ) is $1 / 3$ as much as $e^{x^{3}}$, or $\frac{1}{3} e^{x^{3}}$.

To verify this answer, take its derivative using the chain rule to show that it is indeed the given function. In other words, you should confirm that
to verify that $\frac{1}{3} e^{x^{3}}$ is an antiderivative of $x^{2} e^{x^{3}}$.
According to this discussion, let's now generalize the Undoing the Chain Rule idea by accounting for possible factors and coefficients that will be important in the process.

UNDOING THE CHAIN RULE:

$$
k g^{\prime}(f(x)) f^{\prime}(x) \xrightarrow{\text { Anti-Derivative }} k g(f(x))
$$

Footnote: The coefficient $k$ in this statement of undoing the chain rule is the visible coefficient in the expression $k g(f(x))$ (the antiderivative), but it is not necessarily the visible coefficient in the given rate of change function on the left above. Here $k$ could be expressed as $k_{v} / c$, where $k_{v}$ is the visible coefficient in the given function and $c$ is the constant factor that is part of $f^{\prime}(x)$ from the chain rule. The $c$ that is part of $f^{\prime}$ and the $1 / c$ that is part of $k$ multiply to 1 , leaving $k_{v}$ as the visible coefficient of the rate function.

Before we develop a general method of "undoing the chain rule" to find antiderivatives, the reflection question below gives you a little practice identifying whether or not a rate of change function has the from $k g^{\prime}(f(x)) f^{\prime}(x)$. Notice that, for "undoing the chain rule" to work, one factor in the rate function must be some constant times the derivative of $f(x)$, the argument of $g$.

Reflection Question 9.2.1: Which of the following functions are of the form $\mathrm{kg}^{\prime}(f(x)) f^{\prime}(x)$ ? For the functions that do not have the form $\mathrm{kg}^{\prime}(f(x)) f^{\prime}(x)$, make one small change so that the modified version has this form.
(Remember that multiplication is commutative. The factors of $\operatorname{kg}^{\prime}(f(x)) f^{\prime}(x)$ can appear in any order in the given expression.)
a) $4\left(\sin x^{4}\right) x^{3}$
b) $x^{8} \sqrt{x^{9}+4}$
c) $(\sin x) e^{\sin x}$
d) $\sin ^{3} x \cos x$
e) $\frac{(\ln x)^{5}}{x}$
f) $x e^{x}$
g) $\frac{(\arctan x)^{2}}{1+x^{2}}$
h) $10(\tan (x))^{-7}$

## A Method for Undoing the Chain Rule

Now let's develop a method for applying this idea of undoing the chain rule:

$$
k g^{\prime}(f(x)) f^{\prime}(x) \xrightarrow{\text { Anti-Derivative }} k g(f(x))
$$

After recognizing that the given rate function has the form, the task is to recover .
So, the big question for getting started is:
What should be the focus in the given expression $k g^{\prime}(f(x)) f^{\prime}(x)$, if the desired answer is $k g(f(x))$ ?

## Reflection Question:

If a given rate function has the following structure and factors,

which of $\mathrm{A}, \mathrm{B}$, and C will be the focus to determine the accumulation function $k g(f(x))$ ?
Don't read on until you've thought about this and can give reasoning for your choice.

Hopefully it's obvious to you that ' B ' is the clear choice, because it has all the information needed to find $g(f(x))$. (Notice that ' C ' has no way of revealing what the function $g$ is, and 'A' contains no information about either $f$ or $g$.)

Examples of how to determine where to focus your attention when finding an antiderivatire
The following examples emphasize how to recognize where to focus your attention when trying to find an antiderivative of a rate of change function. We will not find the antiderivatives yet. That will come in the next set of examples.

The given functions have the form $k g^{\prime}(f(x)) f^{\prime}(x)$, making them eligible for undoing the chain rule. For each identify the factor that has the role of $g^{\prime}(f(x))$ and will be the focus in the process of finding the antiderivative.
i) $\quad\left(\sec ^{2} x\right)(\tan x+1)^{5}$

Both factors are composite functions, so the question is: which factor is $f^{\prime}(x)$, the derivative of the argument of the other factor $g^{\prime}(f(x))$ ? Notice that $\frac{d}{d x}(\tan (x)+1)=\sec ^{2} x$, so $\sec ^{2} x$ has the role of $f^{\prime}(x)$, and $(\tan x+1)^{5}$ has the form $g^{\prime}(f(x))$. Thus, $(\tan x+1)^{5}$ will be the focus in finding the antiderivative.
ii) $\frac{7 e^{\sqrt{x}}}{\sqrt{x}}$

Even though this is a rational function (quotient), we can start by rewriting it as the product $7 e^{x^{1 / 2}}\left(x^{-1 / 2}\right)$. Since $e^{x^{1 / 2}}$ is the only composite function in this expression, we can verify that it is has the form $g^{\prime}(f(x))$ by taking the derivative of $x^{1 / 2}$, which is $\frac{1}{2}\left(x^{-1 / 2}\right)$. This is a constant times $x^{-1 / 2}$, the other factor in. So $x^{-1 / 2}$ is associated with $f^{\prime}(x)$, and the factor to focus on in finding the antiderivative is $e^{x^{1 / 2}}$.

## Finding the Antiderivative

Once you have identified the $g^{\prime}(f(x))$ factor of the given function, you can determine the antiderivative, which is of the form $k g(f(x))$. This method can be summarized as:

$$
g^{\prime}(f(x)) \rightarrow k g(f(x))
$$

Comparing these two expressions reveals two steps needed to determine the second from the first:

1) Write a composite function consisting of the antiderivative of $g^{\prime}$ as the exterior, and $f$ unchanged as the argument, i.e. $g(f(x))$. Include any visible constant coefficient from the given function and call this resulting function "the first attempt."
2) Determine, as required, the constant to be multiplied by the first attempt to adjust it resulting in the correct antiderivative. Do this by taking the derivative of the first
attempt, comparing it to the given function, and reasoning with multiplication and/or division.

## Completing the Examples Started Above

Let's apply this technique to find the antiderivatives of the functions in the previous examples.
i) $\quad\left(\sec ^{2} x\right)(\tan x+1)^{5}$

As discussed, the key factor is $(\tan x+1)^{5}$ which has the form $g^{\prime}(f(x))$.
We recognize that the exterior function is "something to the fifth power," i.e. $(f(x))^{5}$. The antiderivative of $x^{5}$ is $\frac{1}{6} x^{6}$, so the first attempt at the antiderivative is $\frac{1}{6}(\tan x+1)^{6}$.

Next to determine if a constant multiple is needed, take the derivative of the first attempt using the Chain Rule:

$$
\frac{1}{6}(\tan x+1)^{6} \xrightarrow{\text { Derivative }} 6 \cdot \frac{1}{6}(\tan x+1)^{6} \cdot \sec ^{2} x
$$

This derivative is equal to the given function, so no multiplier is needed. Thus, the first attempt is correct; the principal antiderivative of is $\frac{1}{6}(\tan x+1)^{6}$.
ii) $\quad \frac{7 e^{\sqrt{x}}}{\sqrt{x}}$

The key factor in this function, as discussed above is $e^{\sqrt{x}}$ or $e^{x^{1 / 2}}$. The first attempt is found by taking the antiderivative of the exterior function " $e$ to the something," leaving the argument $f(x)$ unchanged, and including the coefficient of 7 . So, the first attempt is $7 e^{x^{1 / 2}}$ (since the antiderivative of $e^{x}$ is simply $e^{x}$ ).

To find the constant multiplier, take the derivative of the first attempt using the Chain Rule and compare it to the original function.
$7 e^{x^{1 / 2}} \stackrel{\text { Derivative }}{\rightarrow} 7 e^{x^{1 / 2}} \cdot \frac{1}{2} x^{-1 / 2}$ which can be rewritten as $\frac{7 e^{\sqrt{x}}}{2 \sqrt{x}}$.

The derivative of the first attempt is $1 / 2$ as much as the original function, i.e. there's an unwanted 2 in the denominator. To determine the multiple needed, ask "by what do I need to multiply the first attempt so that the resulting derivative will not have that factor of $1 / 2$ ?" Clearly, we need a multiple of 2. Thus, the finalized antiderivative is

$$
2 \cdot 7 e^{x^{1 / 2}}=14 e^{\sqrt{x}}
$$

You should check that this is indeed the correct antiderivative by taking its derivative to see that it is exactly the original function given.

## More Examples

Example 1 Find the principal antiderivative of $p(x)=x^{4}\left(x^{5}-1\right)^{9}$.

## Solution

First confirm $p$ does qualify for the Undoing the Chain Rule by observing that $x^{4}$ is a constant multiple of the derivative of the argument $x^{5}-1$.

This means that $\left(x^{5}-1\right)^{9}$ has the role of $g^{\prime}(f(x))$.
The first attempt consists of the antiderivative of "something to the 9 th" leaving the argument unchanged. So the first attempt is $\frac{1}{10}\left(x^{5}-1\right)^{10}$.

To check and find a possible constant multiple needed, take the derivative of the first attempt:
$10 \cdot \frac{1}{10}\left(x^{5}-1\right)^{9} \cdot 5 x^{4}$. This is 5 times as much as the original function, so we multiply the first attempt by $1 / 5$ to get the finalized antiderivative:
$\frac{1}{5} \cdot \frac{1}{10}\left(x^{5}-1\right)^{10}=\frac{1}{50}\left(x^{5}-1\right)^{10}$
To confirm this answer, show that its derivative is equal to the original function.

Example 2 Find the principal antiderivative of $d(x)=\frac{\sin x}{\cos ^{4} x}$.

## Solution

Rewrite $d$ as a product: $(\sin x)(\cos x)^{-4}$

Confirm $d$ does qualify for the Undoing the Chain Rule by observing that $\sin x$ is a constant multiple of the derivative of the argument $\cos x$.

This means that $(\cos x)^{-4}$ has the role of $g^{\prime}(f(x))$.
The first attempt consists of the antiderivative of "something to the negative 4th" leaving the argument unchanged. So the first attempt is $-\frac{1}{3}(\cos x)^{-3}$.

To check and find a possible constant multiple needed, take the derivative of the first attempt:
$(-3)\left(-\frac{1}{3}\right)(\cos x)^{-4}(-\sin x)=-(\cos x)^{-4}(\sin x)$. This is -1 times as much as the original function, so we multiply the first attempt by -1 to get the finalized anti-derivativ:

$$
(-1) \cdot\left(-\frac{1}{3}(\cos x)^{-3}\right)=\frac{1}{3}(\cos x)^{-3}
$$

To confirm this answer, show that its derivative is equal to the original function.

Example 3 Find the principal antiderivative of $g(x)=\frac{e^{2 x}}{e^{2 x}+8}$.

## Solution

Rewrite $g$ as a product: $\left(e^{2 x}\right)\left(e^{2 x}+8\right)^{-1}$
Confirm $g$ does qualify for the Undoing the Chain Rule by observing that $e^{2 x}$ is a constant multiple of the derivative of the argument $e^{2 x}+8$.

This means that $\left(e^{2 x}+8\right)^{-1}$ has the role of $g^{\prime}(f(x))$.
The first attempt consists of the antiderivative of "something to the -1 " leaving the argument unchanged. Finding the antiderivative by applying the reverse "power rule" to this composite
function yields $\frac{1}{0} \cdot\left(e^{2 x}+8\right)^{0}$. This is very problematic for more than one reason, saying loudly to us that something's wrong here!

Returning to $\left(e^{2 x}+8\right)^{-1}$, we ask the question "What function has derivative $x^{-1}$ or $1 / x$ ?" You should recall that the function is $\ln x$.

So the first attempt is $\ln \left(e^{2 x}+8\right)$.

To check and find a possible constant multiple needed, take the derivative of the first attempt:
$\left(e^{2 x}+8\right)^{-1} \cdot\left(e^{2 x}\right) \cdot 2$. This is 2 times as much as the original function, so we multiply the first attempt by $1 / 2$ to get the finalized antiderivative:
$\frac{1}{2} \ln \left(e^{2 x}+8\right)$
To confirm this answer, show that its derivative is equal to the original function.

## Accumulation Functions and the Fundamental Theorem

Because of our work in this section and the Fundamental Theorem, we can now write accumulation functions in closed form whenever we're given rate of change functions that have the form of a Chain Rule derivative.

For example, we found that the principal antiderivative of $\left(\sec ^{2} x\right)(\tan x+1)^{5}$ is $\frac{1}{6}(1+\tan x)^{6}$. According to the Fundamental Theorem, that means

$$
\int_{a}^{x}\left(\sec ^{2} t\right)(\tan t+1)^{5} d t=\frac{1}{6}(1+\tan x)^{6}-\frac{1}{6}(1+\tan a)^{6}
$$

In other words, given the rate of change function $f(x)=\left(\sec ^{2} x\right)(\tan x+1)^{5}$, we can easily write down the associated accumulation function in open form: $F(x)=\int_{a}^{x}\left(\sec ^{2} t\right)(\tan t+1)^{5} d t$. But
now because of Undoing the Chain Rule and the Fundamental Theorem, we can find and write this accumulation function $F$ in closed form: $\quad F(x)=\frac{1}{6}(1+\tan x)^{6}-\frac{1}{6}(1+\tan a)^{6}$.

Reflection Question 9.2.2 For each open form accumulation function given: i) write the function in closed form, ii) write in words the meaning of the value represented with function notation, and iii) write a mathematical expression that could be used to compute the represented value with a basic scientific calculator. (Hint: for part i), make use of all the work already done in this section!)
a) $H(x)=\int_{0.9}^{x} \frac{e^{2 t}}{e^{2 t}+8} d t \quad, \quad H(3)$
b) $Q(x)=\int_{\pi}^{x} \frac{\sin t}{\cos ^{4} t} d t \quad, \quad Q(9 \pi / 4)$
c) $\quad F(x)=\int_{-2}^{x} t^{4}\left(t^{5}-1\right)^{9} d t \quad, \quad F(-0.5)$
d) $J(x)=\int_{5.6}^{x} \frac{7 e^{\sqrt{t}}}{e^{\sqrt{t}}} d t \quad, \quad J(5.8)$

## Bacteria Example

A culture bacteria in a particular dish has an initial population of 250 cells and grows at a rate of $r_{B}(t)=50 e^{0.3 t}$ cells / day.
a) Write two expressions for $B$, the population $t$ days after the initial measurement: in open form, and in fully simplified closed form.
b) Represent in 3 different ways the population 10 days after the initial measurement. Find the population value in the most efficient way.
c) Represent the change in population from day 8 to day 15 in open form, with function notation, and find the amount of change.

## Solution

a) Write two expressions for $B$, the population $t$ days after the initial measurement: in open form, and in fully simplified closed form.

The number of bacteria $t$ days after the initial measurement, $B(t)$, consists of the initial value plus the accumulated number of bacteria from 0 to $t$ days after the initial measurement. Thus, the open form representation of $B$ is:

$$
B(t)=250+\int_{0}^{t} 50 e^{0.3 w} d w
$$

To find a closed form for $B$, we'll need to find the antiderivative of the rate function using Undoing the Chain Rule. Since the antiderivative of " $e$ to the something" is simply " $e$ to the something," the first attempt is $50 e^{0.3 w}$, and its derivative is $50 e^{0.3 w} \cdot(0.3)$. This is 0.3 times as much as the original rate function, and so we'll multiply the first attempt by $1 / 0.3$ to get the finalized antiderivative, $\frac{50}{0.3} e^{0.3 w}$. Now apply the Fundamental Theorem to rewrite $B$ :

$$
\begin{aligned}
& B(t)=250+\int_{0}^{t} 50 e^{0.3 w} d w \\
& =250+\left.\frac{50}{0.3} e^{0.3 w}\right|_{0} ^{t} \\
& =250+\frac{50}{0.3} e^{0.3 t}-\frac{50}{0.3} e^{0.3(0)} \\
& =\frac{500}{3} e^{0.3 t}+\frac{250}{3}
\end{aligned}
$$

NOTE!! Plugging $t=0$ into the antiderivative results in a NON-ZERO value, $-50 / 0.3$. Thus the constant term in the final answer is $250-50 / 0.3=250 / 3$. It is essential that you fully apply the Fundamental Theorem and include this term for which $t=0$ to get the correct closed form for $B!!$

Thus, the closed form representation of $B$ is $B(t)=\frac{500}{3} e^{0.3 t}+\frac{250}{3}$
b) Represent in 3 different ways the population 10 days after the initial measurement. Find the population value in the most efficient way.

The population after 10 days can be represented with...
... function notation: $B(10)$
...the open form of the accumulation function: $250+\int_{0}^{10} 50 e^{0.3 w} d w$
...the closed form of the accumulation function: $\frac{500}{3} e^{0.3(10)}+\frac{250}{3}$

The closed form of the accumulation function most easily generates the number of bacteria:
$\frac{500}{3} e^{0.3(10)}+\frac{250}{3}=3430$ bacteria (Don't round up to 3431 since that number of bacteria has not yet been reached at exactly 10 days after the initial measurement.)
c) Represent the change in population from day 8 to day 15 in open form, with function notation, and find the amount of change.

The change in population from day 8 to day 15 can be represented in function notation as $B(15)-$ $B(8)$.

We could use the open form of $B$ to express this value, but it is rather lengthy:

$$
\left(250+\int_{0}^{15} 50 e^{0.3 w} d w\right)-\left(250+\int_{0}^{8} 50 e^{0.3 w} d w\right)
$$

Alternatively, we can consider a different function: the net change in population starting from $t=$ 8 , rather than from $t=0$. Then the change in bacteria population over this time period would be represented much more compactly as $\int_{8}^{15} 50 e^{0.3 w} d w$.

To find this value, apply the Fundamental Theorem to this new integral, using the antiderivative already found:

$$
\int_{8}^{15} 50 e^{0.3 w} d w=\left.\frac{50}{0.3} e^{0.3 w}\right|_{8} ^{15}=\frac{500}{3} e^{0.3(15)}-\frac{500}{3} e^{0.3(8)}=13,165 \text { bacteria }
$$

Note this value is not the number of bacteria at $t=15$ days but is rather the net change, or added number of bacteria, starting from $t=8$ to $t=15$ days since the initial measurement.

## Exercise Set 9.2

1. For each accumulation function given in open form, find an equivalent closed form.
a) $\int_{4}^{x} \frac{(\arcsin t)^{4}}{\sqrt{1-t^{2}}} d t$
b) $\int_{1 / 2}^{x} \frac{(\ln t)^{6}}{t} d t$
c) $\int_{-2}^{x} t^{6} e^{t^{7}} d t$
d)
$\int_{-2}^{x} 12\left(\sin ^{2} t\right)(\cos t) d t$
e) $\int_{5}^{x} \frac{d t}{2-8 t}$
f) $\int_{13}^{x} \frac{d t}{t \sqrt{\ln t}}$
g) $\int_{2 \pi}^{x} \frac{\cos t}{8 \sin t+7} d t$
h) $\int_{-1}^{x} \frac{\cos (5 / t)}{t^{2}} d t$
i) $\int_{-2}^{x} t^{2} \sqrt{9+t^{3}} d t$
j) $\int_{4 / 3}^{x} \frac{d t}{t(1+\ln t)}$
k) $\int_{0}^{x} 2^{9 t} d t$
1) $\int_{\pi / 2}^{x} \sin (7 t) \cos (7 t) d t$
m) $\int_{0}^{x} \frac{1}{(6 t+9)^{1.8}} d t$
n) $\int_{16}^{x}\left(t^{2}+10 t+32\right)^{3}(t+5) d t$
o) $\int_{3 \pi / 8}^{x} \tan t \sec ^{2} t d t$
p) $\int_{15}^{x} \frac{t}{t^{2}+6} d t$
2. Some rate functions require algebraic manipulation or simplification to set the stage for Undoing the Chain Rule or other antiderivative techniques.

Find an equivalent closed form for each function.
a) $\int_{\pi / 4}^{x} \frac{5 t+4}{t^{2}+1} d t \quad$ (Hint: begin by writing the rate function as a sum of fractions)
b) $\int_{1}^{x} 4 \tan t d t \quad$ (Hint: begin by using a trig identity to change the form of the rate function)
3. The goal is efficiency....not the procedure!

We introduced the step-by-step method for Undoing the Chain Rule only as a means of getting started in finding these types of antiderivatives. Instead of memorizing these steps and following them for every future problem, the real goal is that eventually you can all at once imagine the
function whose derivative (found by the chain rule) is the function you are given. The obvious benefit of learning this skill is having maximum efficiency in finding such accumulation functions in closed form - there is no faster way! In addition, learning and practicing an "all at once" technique keeps you focused on the Chain Rule rather than some new and unrelated method. It will also sharpen your mental computation skills.

Review your solutions to \#1 above, thinking about the common overall pattern to these solutions rather than the procedure. Then try to determine the antiderivatives of the functions below in your head; that is, try to write down only the final answer. Do your best, but don't be afraid to go for it!... because remember with any attempt, you can take its derivative to check it and then adjust if you need to.

For each function, write down, without any intermediate steps, its principal antiderivative. Take the derivative of your answer to confirm it's correct; if it isn't then make an adjustment so that the new version is correct.
a) $\cos (-14 x)$
b) $\left(x^{2}-9\right)^{5} x$
c) $\frac{(\arctan x)^{2}}{1+x^{2}}$
d) $(\sin x) e^{\cos x}$
e) $\frac{\sqrt{e^{-x}+5}}{e^{x}}$
f) $\frac{1}{\sqrt{6 x}}$
g) $\left(\sin 2 x^{7}\right) x^{6}$
h) $12 e^{-3 x}$
4. Application Problem
[Note to Pat: I considered putting this summary / review of the steps at the end of the section right before the Exercise Set, but I think I prefer to leave it out.]

## Summary of Undoing the Chain Rule Method

1) Verify that the rate function indeed has the form $k g^{\prime}(f(x)) f^{\prime}(x)$
2) Find and focus in on the key factor in your rate function: $g^{\prime}(f(x))$. (This can be anywhere in the expression - beginning, middle, or end - since multiplication is commutative.)
3) Determine the antiderivative of the exterior function $g^{\prime}$, and write it (i.e. $g$ ) with argument $f$ unchanged. This composite function, multiplied by the coefficient of the given rate function, is the 'first attempt.'
4) Check the first attempt by finding its derivative with the Chain Rule. If the derivative of the first attempt is equal to the original rate function, then you're done; the first attempt is the principal antiderivative in closed form.
5) Otherwise, the derivative should be some constant multiple $c$ times the original rate function (if it's not, you've made an error somewhere in steps 1-4 above). Multiply the first attempt by $1 / c$ to produce the correct antiderivative.

Because the mindset in finding accumulation functions is always "what function has my given function as its derivative?", you should continually be taking derivatives of your proposed answers to check if your antiderivative is correct. This verification process will play an essential role in all of the methods in this chapter, including undoing the chain rule.

