Kevin Moore

CHAPTER 1: INTRODUCTION

This proposal describes an anticipated investigation into student trigonometry and trigonometric function understandings. Trigonometry and trigonometric functions have been an important part of the high school and undergraduate mathematics curriculum for the past century. In addition to the numerous mathematical topics (e.g., Fourier series and integration techniques) related to trigonometric functions, various topics of the sciences (e.g., projectile velocity and wave behavior) are reliant on trigonometric functions. Trigonometry and trigonometric functions also offer one of the earlier mathematical experiences that combine geometric, algebraic, and graphical reasoning with functions that cannot be computed through algebraic computations. Though trigonometry has been a part of the mathematical curriculum for over a century, it frequently is the case that students and teachers have difficulty reasoning about topics reliant on trigonometric function understandings (Brown, 2005; Fi, 2003; Thompson, Carlson, & Silverman, 2007; Weber, 2005). To further complicate things, few studies have investigated the reasoning abilities and understandings needed to understand and use trigonometric functions.

Student difficulties relative to trigonometry and trigonometric functions may be related to the approach that curricular materials take when introducing and developing students' trigonometric function understandings. In most US textbooks, trigonometric functions are often introduced in multiple contexts (e.g., triangle trigonometry and unit circle trigonometry). In each of these contexts, trigonometric functions are presented as being used for different purposes (e.g., determining the side of a triangle or finding a coordinate). It is also the case that mathematics curriculum typically treats these trigonometries as unrelated or only slightly related. This may hinder students from developing coherent trigonometric function understandings that contain strong connections across the various trigonometric contexts.

When speaking of the various contexts of trigonometry being only slightly related, this is not intended to imply that curriculum does not use one context as an introduction to another. For instance, triangle trigonometry is often used as a lead into unit circle trigonometry. When stating that the various contexts are often treated as being slightly related, I intend to mean that common foundations to each context, such as angle measure, are not leveraged to develop coherence between each trigonometry. As Thompson (2008) has recently argued, we should consider building on meanings and foundations that are common between the various trigonometries in order to promote coherent student understandings. The consideration of building on common meanings and foundations between the trigonometries is a major focus of this proposed investigation.

This proposal describes an anticipated investigation that attempts to build on common foundations between the various trigonometry contexts to develop student understandings. As mentioned, limited research has been conducted with regards to student reasoning upon the introduction of trigonometric functions. The proposed study attempts to fill this gap by investigating the developing understandings of students during the introduction of trigonometric functions in an undergraduate precalculus course. Furthermore, the proposed investigation includes a description of the instructional sequence to be used. Supporting the development of student understandings of angle measure, the use of the radius as a unit of measurement, and the unit circle are also focal points of the instructional goals. As a result, this investigation should also offer insights about student understandings of concepts foundational to the study of trigonometry and trigonometric functions.

What follows in this chapter is a brief introduction of the research problem and research questions that frame the study. This first chapter also discusses various foundational reasoning abilities that will form the basis for the understandings deemed necessary for the topics of trigonometry. During the discussion of these foundational reasoning abilities, I briefly address the role these abilities play in constructing trigonometric function understandings. The role of these foundational reasoning abilities is then connected to findings from the research literature on student and teacher trigonometric understandings (Chapter 2). Chapter 3 presents the methodology of the proposed study and includes discussing the body of ideas that are entailed in understanding trigonometric functions in the context of examining student thinking during my pilot study. Chapter 4 then revisits, in greater detail, connections between the foundational reasoning abilities described and developing student understandings of trigonometric functions in each trigonometry context. In light of this analysis, a description of a portion of the proposed instructional sequence is presented.

Statement of the Problem

Although trigonometry has been a part of the United States mathematics curriculum for over a century and the NCTM Standards have called for connections between the various trigonometry contexts (NCTM, 1989, 2000), trigonometry has remained what is considered a highly difficult topic for students (Brown, 2005; Thompson, 2008; Weber, 2005). While trigonometry is a seemingly important and difficult topic in mathematics, very limited attention has been given to trigonometry in mathematics education research and curriculum development (Brown, 2005; Weber, 2005). The attention that has been given to trigonometry has commonly revealed that student and teacher understandings that are foundational to trigonometric function understandings need more focus and that coherence between the multiple contexts of trigonometry needs to be developed (Brown, 2005; Thompson, 2008; Thompson, et al., 2007; Weber, 2005).

The difficulties that students encounter in developing coherent trigonometric understandings is likely multifaceted. First, trigonometric functions require sophisticated reasoning relative to function as a process. Trigonometric functions are often one of a student's initial experiences with a function that cannot be computationally evaluated. Hence, reasoning about trigonometric functions relies on reasoning about a function in a manner that one can imagine input values to a function being evaluated and output values being produced without actually performing computations. Although the ability to conceptualize a function as a process has been shown to be a difficult function conception for students (Carlson & Oehrtman, 2004; Harel & Dubinsky, 1992; Oehrtman, Carlson, &

Thompson, 2008; Sierpinska, 1992; Thompson, 1994b), trigonometric functions offer the opportunity to develop and promote this reasoning in students. Yet, it does not appear that the current mathematics curriculum addresses or develops this foundational understanding and way of reasoning.

Trigonometry may also be a difficult topic for students due to an incoherence of previous understandings and a lack of developed reasoning abilities that are foundational to building understandings of trigonometry. For instance, it is necessary that student images of angle measure and function be developed previous to an introduction of trigonometric functions such that these images support the student construction of coherent and connected understandings of trigonometric ideas.

This investigation attempts to put these considerations and conjectures into action in order to contribute to the limited body of research literature on the development of student trigonometry and trigonometric function understandings. The insights gained about how student understandings and reasoning abilities develop will also inform the future design of trigonometry curriculum.

Research Questions

This proposed study investigates student trigonometric understandings. The primary research question driving this study is:

- What understandings of trigonometric functions do students develop during a trigonometry instructional sequence that is designed on the foundations of quantitative and covariational reasoning?

Supporting research questions derived from the theoretical foundation and design of the study include the following:

- What role does quantitative reasoning and covariational reasoning play in developing student understandings of trigonometric functions?
- What understandings of the topics foundational to trigonometric functions (e.g., angle measure, radian as a unit of measurement, and the unit circle) do students develop during the trigonometry instructional sequence?
- How do the foundational understandings of angle measure, radian, and unit circle influence student understanding of trigonometric functions?
- How do student trigonometric function conceptions influence student reasoning in the trigonometric contexts of the unit circle and right triangles?

Motivation for the Study

The motivation for conducting this study is influenced by my own personal experiences with trigonometry and trigonometric functions. Trigonometry was one of the few topics of mathematics that left me feeling unsatisfied as I completed my mathematical courses. Even though I had high success with courses reliant on trigonometric functions, I never felt I truly understood trigonometric functions or their role in mathematics. I was left pondering if there was any relationship between the application of trigonometric functions with right triangles, circles, and periodic relationships. I believed the unit circle

Luis Saldanha

INTRODUCTION

This dissertation explores the emergence of ideas among eight high school students as they participated in a classroom teaching experiment addressing statistical concepts. Instruction aimed to move students toward embedding statistical inference within the foundational idea of *sampling distributions*—the distributional structure of a collection of a sample statistic's values that one conceives as emerging in the long run, under repeated random sampling. This embedding is deep and the connections are important, yet both are rarely a subject of instruction in statistics. In contrast, instruction in this teaching experiment engaged students with the inner logic of statistical inference, pushing them to explore its deep structure.

Given the study's exploratory nature, the central aim of this dissertation is to gain insight into what students understood of the ideas addressed in instruction in relation to their engagement with that instruction. The method employed to develop such insight is the *retrospective analysis* (Cobb, 2000; Steffe & Thompson, 2000). That is an after-the-fact systematic and coordinated examination of the data generated in the teaching experiment.[1] In this particular case, my examination is directed at characterizing students' understandings in terms of plausible underlying imagery and conceptual operations.

My examination of these data takes a multi-pronged approach: I characterize instructional activities, instructional interactions and student engagements, and students' emergent and stable understandings. Moreover, my analyses attempt to capture the emergent and dynamic interplay among these components. This results in a characterization of the teaching experiment as a sequence of interrelated instructional activities unfolding in synergy with the emergence of students' ideas.

The dissertation is divided into two broad parts distinguished by their content and structure. Chapters I through IV constitute the first part. Chapter I develops a directed analysis of previous research, highlighting issues of relevance for this study. Chapter II provides the background and context for the teaching experiment, detailing its important aspects and situating it within the prior relevant research. Chapter III elaborates the theoretical lenses employed in my analyses, taking into consideration the perspectives that underlay the design and implementation of the

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[1] My role as research assistant in conducting this teaching experiment (NSF Grant No. REC-9811879; P.I. Patrick Thompson) enabled me to position myself as a co-investigator in this analysis. The narrative in this dissertation is thus spun from the perspective of a member of the research team attempting to communicate with nonmembers.

teaching experiment. Chapter IV details the procedures and methods employed in my analyses of the data.

In the dissertation's second part analyses and results are distributed across Chapters V through IX. Instructional interactions are characterized as unfolding in a sequence of four interrelated phases and students' conceptions that emerged within them are detailed. Each of Chapters V through VIII addresses one distinct phase of instruction. Chapter IX gives a summary overview of the teaching experiment and elaborates conclusions.

Looking ahead, much of this study focuses on students' experiences and thinking as they worked with collections of a sample statistic's values, organizing them in ways that could provide a basis for making statistical inferences. The study indicates that in exploring the deep structure of statistical inference, students *grappled* with that structure. Conceiving a collection of a sample statistic's values as a sampling distribution entails the coordination of multiple objects and actions in a hierarchical structure. Students experienced significant difficulties developing this coordination, even when individual objects and actions seemed unproblematic for them to envision. This suggests that developing a coherent understanding of sampling distributions is non-trivial and may be rooted in abilities to navigate with facility among hierarchically structured objects and processes.

Yan Liu

CHAPTER I

STATEMENT OF PROBLEM

Teachers' understanding of significant mathematical ideas has profound influence on their capacity to teach mathematics effectively (Thompson 1984; Ball and McDiarmid 1990; Ball 1990; Borko, Eisenhart et al. 1992; Eisenhart, Borko et al. 1993; Simon 1994; Thompson and Thompson 1996; Sowder, Philipp et al. 1998; Ball and Bass 2000), and, in turn, on what students end up learning and how well they learn (Begle 1972; 1979). To elaborate, first, teachers' personal understanding of mathematical ideas constitutes the most direct source for what they intend students to learn, and what they know about ways these ideas can develop. Second, how well teachers understand the content they are teaching have critical influence on their pedagogical orientations and their ability to make instructional, curricular, and assessment decisions (Thompson 1984; McDiarmid, Ball et al. 1989; Borko, Eisenhart et al. 1992; Dooren, Verschaffel et al. 2002). This ensemble of teachers' knowledge (Shulman 1986), orientations (Thompson, Philipp et al. 1994), and beliefs (Grossman, Wilson et al. 1989)—of mathematical ideas, and of ways of supporting students' learning of these ideas, plays important roles in what students can learn and how well they learn in the instructional settings.

This has important implications for how teacher educators think about ways of supporting teachers' professional development. That is that, supporting transformation of teaching practices takes careful analysis of teachers' personal and pedagogical understanding. Such efforts increase the likelihood that what teachers teach and how they teach have the potential of supporting students to develop coherent and deep understanding of mathematics.

Probability and statistical inference are among the most important and challenging ideas that we expect students to understand in high school. Probability and statistical inference have had an enormous impact on scientific and cultural development since its origin in the mid-seventeen century. The range of their applications spread from gambling problems to jurisprudence, data analysis, inductive inference, and insurance in eighteen century, to sociology, physics, biology and psychology in nineteenth, and on to agronomy, polling, medical testing, baseball and innumerable other practical matters in twentieth (Gigerenzer, Swijtink et al. 1989). Along with this expansion of applications as well as the concurrent modification of the theories themselves, probability and statistical inference have shaped modern science, transformed our ideas of nature, mind, and society, and altered our values and assumptions about matters as diverse as legal fairness to human intelligence. Given the extraordinary range and significance of these transformations and their influence on the structure of knowledge and power, and on issues of opportunity and equity in our society, the question of how to support the development of coherent understandings of probability and statistical inference takes on increased importance.

Since 1960s, there have been abundant research studies conducted to investigate ways people understand probability and statistical inference. Psychological and instructional studies consistently documented poor understanding or misconceptions of these ideas among different population across different settings (Kahneman and Tversky 1973; Nisbett, Krantz et al. 1983; Konold 1989; 1991; Konold, Pollatsek et al. 1993a;

Fischbein and Schnarch 1997). Contrary to the overwhelming evidences of people's difficulties in reasoning statistically, there is in general a lack of insight into what is going on in the transmission of this knowledge in classroom settings. Particularly, research on statistics education has attended to neither teachers' understanding of probability and statistics, nor to their thinking on how to teach these subjects (Truran 2001; Garfield and Ben-Zvi 2003).

The goal of this dissertation study is to explore teachers' personal and pedagogical understanding of probability and statistical inference. To this end, our research team undertook a teaching experiment[1] with nine high school mathematics teachers in the context of a professional development seminar. This teaching experiment is an early step of a bigger research program, which aims to understand ways of supporting teachers learning and their transformations of teaching practices into one that is propitious for students learning in the context of probability and statistics. As a precursor, this teaching experiment is highly exploratory. The research team designed the teaching experiment with the purpose of provoking the teachers to express and to reflect upon their instructional goals, objectives, and practices in teaching probability and statistics. The primary goal was to gain an insight into the issues, both conceptual and pedagogical, that teachers grapple with in order to teach probability and statistics effectively in the classroom.

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[1] This teaching experiment is part of a five-year, longitudinal research project "An investigation of multiplicative reasoning as a foundation for teaching and learning stochastic reasoning," designed and directed by Dr. Patrick Thompson, my dissertation advisor and professor of mathematics education at Vanderbilt University. Since I joined the research team 5 years ago, I have been integrally involved in all of its facets: instructional design, data collection, organization, and interpretation.

This dissertation will present a retrospective analysis of this teaching experiment. Specifically, the aims of this dissertation are:

1) To construct an explanation of teachers' personal and pedagogical understanding of probability and statistical inference;
2) To create a theoretical framework for constructing such an explanation.

To explicate my research purposes, let me first explain what I mean by "understanding" and the method I use in developing descriptions of an understanding. By "understanding" I follow Thompson & Saldanha (2002) to mean that which "results from a person's interpreting signs, symbols, interchanges, or conversation—assigning meanings according to a web of connections the person builds over time through interactions with his or her own interpretations of settings and through interactions with other people as they attempt to do the same." Building on earlier definitions of understanding based on Piaget's notion of assimilation, e.g. "assimilating to an appropriate scheme" (Skemp 1979), Thompson & Saldanha (*ibid.*) extend its meaning to "assimilation to a scheme", which allows for addressing understanding people do have even though it could be judged as inappropriate or wrong. As a result, they suggested that a description of understanding require "addressing two sides of the assimilation—what we see as the thing a person is attempting to understanding and the scheme of operations that constitutes the person's actual understanding." (*ibid.*, p. 11)

To construct a description/explanation of a person's understanding, I adopt an analytical method that Glasersfeld called conceptual analysis (Glasersfeld 1995), the aim of which is "to describe conceptual operations that, were people to have them, might result in them thinking the way they evidently do." Engaging in conceptual analysis of a person’s understanding means trying to think as the person does, to construct a conceptual structure that is isomorphic to that of the person. This coincides with the notion of *emic* perspective in the tradition of ethnographic research, i.e., the “insider’s” or “native’s” interpretation of or reasons for his or her customs/beliefs, what things mean to the *members of a society*, as opposed to *etic* perspective: the external researcher's interpretation of the same customs/beliefs. In conducting conceptual analysis, a researcher builds models of a person’ understanding by observing the person’ actions in natural or designed contexts and asking himself, “What can this person be thinking so that his actions make sense from his perspective?” (Thompson 1982) In other words, the researcher/observer puts himself into the position of the observed and attempt to examine the operations that he (the observer) would need or the constraints he would have to operate under in order to (logically) behave as the observed did (Thompson 1982).

As a researcher engage in the activity of constructing description /model /explanation (henceforth explanation) of his subjects’ understanding, he should in the mean time subject his very activity to examination, i.e., to reflectively abstract (Piaget 1977) the concepts and operations that he applies in constructing explanations. When the researcher becomes aware of these concepts and operations, and can relate one with another, he has an explanatory/theoretical framework, which usually opens new possibilities for the researcher who turns to using it for new purposes (Steffe and Thompson 2000). There is a dialectic relationship between these two kinds of analyses—constructing explanations of a person’ understanding and creating a theoretical framework for constructing such explanations. The theoretical framework and the explanations exert a reciprocal influence upon each other as they are simultaneously constructed. Theoretical framework is used in constructing explanations of understandings. As one refines the understandings, the appearance of the framework changes, as one refines the framework, the understandings may be modified (Thompson 1982).

It is important to note that a theoretical framework does not emerge entirely from the empirical work of trying to understand a person's actions and thinking. It could draw upon theoretical constructs established in an earlier conceptual analysis, or informed by others' work in the existing literature. And most often it is heavily constrained/enabled by the epistemology or background theories that the researcher embraces in his work (e.g. Thompson 1982). In the following chapters, I will first present a review of relevant literature with the purpose of highlighting the theoretical constructs that might potentially constitute part of the framework. This first part of this review presents a historical and conceptual analysis of probability and statistical inference. The second part reviews existing research on ways people/students understand probability and statistical inference, and the difficulties they experience as they learn these ideas. My goal of this review is to provide a vantage point for understanding teachers' knowledge and to highlight a way of understanding these ideas that are grounded in meanings and making connections amongst these ideas.

In Chapter 3, I will present the background theories and methodologies that guide the conceptualization of my research questions and the design and implementation of the teaching experiment. Chapter 4 is a conceptual analysis of the probability, hypothesis testing, and margin of error. In Chapter 5, I will first provide an overview of the teaching experiment. Following this, I will sketch the background of this teaching experiment by summarizing the prior teaching experiments we conducted with high school students. Last, I will provide a detailed description of the seminar by summarizing the daily activities and interviews, as well as the themes that we intended to emerge. Chapter 6, 7, and 8 are each devoted to a particular set of ideas: probability, hypothesis testing, variability and margin of error.

CHAPTER II

LITERATURE REVIEW

2.1 UNDERSTANDING PROBABILITY AND STATISTICAL INFERENCE: A HISTORICAL AND CONCEPTUAL PERSPECTIVE

My investigation of teachers' understanding in probability and statistical inference is motivated by the purpose of supporting the development of students' understanding by improving teacher education in this subject area. This study not only has to be built upon a knowledge of students and teachers' understanding from existing literature and prior research, but also an appreciation of the many ways probability and statistical inference are understood historically.

The development of the theories of probability and statistical inference has been riddled with controversy. For example, the concept of probability is often used to refer to two kinds of knowledge: *frequency-type probability* "concerning itself with stochastic laws of chance processes," and *belief-type probability* "dedicated to assessing reasonable degrees of belief in propositions quite devoid of statistical background" (Hacking 1975 p.

Jason Silverman

CHAPTER I

INTRODUCTION

This study investigates conditions that enable teachers to teach mathematics for understanding. It is not well understood how pedagogical or mathematical knowledge developed in university courses exhibits itself in pre-service teachers' (PSTs') classroom practices.

My aim is to understand the intricacies of this "transfer" from a university setting to school-based teaching practices. In doing this, I focus first on PSTs' understandings of mathematics as the primary resource upon which they draw while teaching. The importance of teachers' knowledge of content has been acknowledged by a variety of scholars (Ball, 1993; Ball & McDiarmid, 1989; Bransford, Brown, & Cocking, 2000; Grossman, 1990; Grossman, Wilson, & Shulman, 1989; Ma, 1999; Schifter, 1990, 1995; Shulman, 1986). However, it is axiomatic that a teacher's knowledge of mathematics alone is insufficient to support his or her attempts to teach for understanding. In that vein, Shulman (1986) coined the phrase *pedagogical content knowledge* [PCK], or specific content knowledge as applied to teaching, to address what at that time had become increasingly evident – that content knowledge itself is not sufficient for teachers to be successful. Ma (1999) and Stigler and Hiebert (Stigler & Hiebert, 1999) further refined the idea of PCK by arguing that teachers need a *profound* understanding of mathematics – knowledge having the characteristics of breadth, depth, and thoroughness: "Breadth of understanding is the capacity to connect a topic with topics of similar or less conceptual power. Depth of understanding is the capacity to connect a topic with those of greater conceptual power. Thoroughness is the capacity to connect all topics" (p. 124).

My first research question builds from Ma's construct of profound understanding of mathematics:

Research Question 1:

What understandings of function will a group of PSTs have after participating in instruction that employs simultaneous covariation of quantities as a pathway to their development of profound understandings of function?

My previous work with student teachers (Silverman, 2004a) led me to believe that PST's naïve conceptions of "profound" understandings of mathematics are inconsistent with teaching mathematics for understanding. Since teachers' understandings of mathematics enable or constrain their ability to orchestrate mathematical discussions that provide students with opportunities to make sense of advanced mathematical ideas, it is important for teacher educators to understand both the understandings with which PSTs enter our programs and ways in which those understandings can be productively influenced. By *teachers' understandings of mathematics* I mean "the loose ensembles of actions, operations, and ways of thinking that come to mind unawarely – of what they wish their students to learn, and the language in which they have captured those images" (Thompson & Thompson, 1996, p. 16). It is against the background of the images that teachers hold with regard to their own understandings and of the understandings they hope students will have that they select tasks, pose questions, and make other pedagogical decisions.

Thus, in this study, I am extending Ma's (1999) notion of profound mathematical understanding in two key ways. First, I argue that teachers must develop explicit images of (1) the mathematics that they want their students to understand, (2) an understanding of the pedagogical importance of these understandings, and (3) a sense of how these understandings might develop in students. Second, I will build upon Simon's (2002) notion of a *key developmental understanding* of a mathematical idea to propose the idea of *key pedagogical understanding* of a mathematical idea as a threshold for teaching for mathematical understanding. A *key developmental understanding* is a particular understanding of a mathematical idea that facilitates understanding a variety of additional mathematical topics. A *key pedagogical understanding* involves an individual's awareness of the pedagogical implications of those key developmental understandings of important mathematical ideas. While teacher education research locates notions such as profound understanding of fundamental mathematics, key developmental understandings, and pedagogical content knowledge as particular states along a developmental trajectory, I will argue that focusing on the idea of key pedagogical understanding addresses how one might one come to develop such understandings. As such, I propose a second broad research question:

Research Question 2

How do PSTs' understandings of covariation impact their image of instruction and their engagement with students when teaching concepts of function? Put another way, in what ways can a profound understanding of covariation serve as a key pedagogical idea in teaching for understanding of functions?

Carlos Castillo-Garsow

STATEMENT OF THE PROBLEM

The NCTM 2000 principles and standards for 9-12th graders includes both "use mathematical models to represent and understand quantitative relationships" (p. 303) and analyze change in various contexts" (p. 305). These two ideas are intimately related. In order to analyze changes, students must understand the situation as involving quantities that are changing. When a student interprets a situation in terms of examining quantities and the relationships between those quantities, that student is engaged in mathematical modeling. Conversely, a substantial amount of the mathematical work in science and engineering involves the study of dynamical systems: mathematical models of changing quantities. The NCTM standards also point to the importance of students' understanding exponential change as part of "Analyzing changes in various contexts." This touches specifically on dynamical systems modeling, as understanding exponential functions is core to understanding both difference and differential equation modeling.

Several researchers have studied ways in which students might operationalize reasoning about relationships between changing quantities, called covariational reasoning -- see: Confrey and Smith, 1994, 1995; Thompson, 1994a; Saldanha and Thompson 1998; Carlson, Jacobs, Coe, Larsen, and Hsu, 2002; Thompson, 2008, 2008MS. However, these researchers have come to very different meanings of covariation and these different meanings result in different connections among covariational reasoning, constant rate of change, and exponential reasoning. (Confrey and Smith, 1994, 1995; Thompson, 2008). It is important to flesh out these differences because students who use the mental operations described by these authors will approach and understand function and rate in very different ways.

The purpose of this study is to examine the ways in which students operationalize covariation, the consequences of students' covariational reasoning for exponential reasoning, and their subsequent understandings of dynamical systems modeling.

Over the past two and a half years, the Teachers Promoting Change Collaboratively (TPCC) research project has engaged in a classroom intervention, following students as they progress as a class Algebra I, Geometry, and Algebra II classes. The researchers have worked in collaboration with the teachers of these classes on a curriculum that emphasizes covariational reasoning and mathematical modeling. Originally, the purpose of this intervention was to provide cases of teachers engaged in instruction to be used for teacher professional development; however, this environment also provides an ideal opportunity to study the meanings of covariation the students have developed as a result of their interactions with the course.

Specifically, this project will examine the consequences of those meanings for exponential reasoning and mathematical modeling by engaging the students in challenging problems: the development of the exponential model from first-principles (Smith, Haarer, and Confrey, 1997), and an investigation of the Verhulst model – one of the earliest adaptations of the exponential model in population modeling (Verhulst 1838/1977). The Verhulst model was selected because it is commonly used as first model for introducing ideas of population modeling (Kingsland, 1982; Edelstein-Keshet, 1988; Begon, Harper, and Townsend, 1996; Brauer and Castillo-Chavez, 2000).

Simply put, the questions that I hope to answer are: when engaged in instruction that emphasize covariational reasoning, what meanings of covariation do students actually develop, and what are the consequences of those meanings for students' understanding of what is conventionally taken as dynamical systems modeling?

Scott Courtney

CHAPTER I - STATEMENT OF PROBLEM

In *The Teaching Gap* (1999), Stigler and Hiebert, drawing on the conclusions of the Third International Mathematics and Science Study (TIMSS), highlighted the necessity for reform in mathematics education in the United States. In the years since *The Teaching Gap* there has been one generally agreed upon theme – that students are not developing a satisfactory level of mathematical proficiency (e.g., Baldi, Jin, Skemer, Green, & Herget, 2007; Gonzales, Calsyn, Jocelyn, Mak, Kastberg, Arafeh, Williams, & Tsen, 2000; Gonzales, Guzman, Partelow, Pahlke, Jocelyn, Kastberg, & Williams, 2004).

Although the mathematics performance of elementary and secondary students has shown some improvement over the past decade on tests such as the National Assessment of Educational Progress (NAEP), this improvement has not been in all grades assessed and is not equal for all groups of students (Hall & Kennedy, 2006; NCES, 2007; Baldi, Jin, Skemer, Green, & Herget, 2007). Several documents have indicated the important role that teachers play, not only in what students learn, but also in any mathematics reform effort (e.g., U.S. Department of Education, 2008; Borasi & Fonzi, 2002; NCTM, 2000; Lampert, 2001; Stigler & Hiebert, 1999; Wenglinsky, 2002; College Board, 2006). Therefore, in order to most propitiously influence student learning, characterizations of effective teachers and effective teaching would seem of paramount importance. Furthermore, once identified, the ability to promote these characteristics in both pre-service and in-service teachers would be highly desirable.

In the past two decades a number of researchers have attempted to identify and explore the characteristics of mathematics teaching and teachers. Several researchers have investigated teacher beliefs (e.g., Cooney, 1985; Ernest, 1998; Leder, Pehkonen, & Torner, 2002; Thompson, A. G., 1984, 1992); others have placed a focus on identifying the structure of teacher knowledge in an attempt to identify a knowledge base for teaching (e.g. Ball, 1988; Ball & Bass, 2000; Ball, Hill and Bass, 2005; Rowland, Huckstep, & Thwaites, 2005; Shulman, 1986, 1987). In addition, various researchers have commented on the difficulty inherent in attempting to distinguish between knowledge and beliefs (Thompson, A. G., 1996; Fennema & Franke, 1992; Grossman, Wilson, & Shulman, 1989).

According to Ball (2003), "The quality of mathematics teaching and learning depends on what teachers do with their students, and what teachers can do depends on their knowledge of mathematics." (p. xv-xvi). Although several studies have shown that a teachers' content knowledge influences their teaching practices (Fernández, 1997; Sowder, Philipp, Armstrong, & Schappelle., 1998; Swafford, Jones, & Thornton, 1997), attempts to quantify a link between teacher subject matter knowledge and student achievement has been largely inconclusive (Monk, 1994; Rowan, Chiang, & Miller, 1997; Begle, 1979; Eisenberg, 1977).

According to Thompson and Thompson (1996), "The most plausible explanation for these puzzling results is that the two variables, teacher subject matter knowledge and student learning, were inadequately conceptualized…[and] that teacher and student knowledge and the relationship between the two are complex theoretical constructs that defy simplistic definitions and conceptualizations" (p. 2). Ball (2003) asserts that although it is "widely agreed among the mathematics education community that effective mathematics teaching depends on teachers' knowledge of content, the nature of the knowledge required for such teaching is poorly specified, and the evidence concerning the nature of the mathematical knowledge that is needed to improve instructional quality is surprisingly sparse" (p. xvi).

Beginning with Shulman's (1986, 1987) seminal work regarding pedagogical content knowledge (PCK), several researchers have focused on the form, nature, organization, and content of teachers' mathematical knowledge (Ball, 1990; Lampert, 1991; Leinhardt & Smith, 1985; Marks, 1987; Steinberg, Marks, & Haymore, 1985; Wilson, Shulman, & Richert, 1987; Thompson, 1984; Thompson, Philipp, Thompson, & Boyd, 1994). In the view of Thompson and Thompson (1996), "This work has highlighted the critical influence of teachers' mathematical understanding on their pedagogical orientations and decisions — on their capacity to pose questions, select tasks, assess students' understanding, and make curricular choices" (p. 2). According to Schoenfeld (2000), "Such studies indicate ways in which teachers' knowledge shapes what the teachers are able to do in the classroom - at times constraining their options, at times providing the support-structure for a wide range of activities" (p. 247).

In 1996, Thompson and Thompson coined the term "mathematical knowledge for teaching (MKT)" to refer to understandings that "cut across the types of knowledge typically embraced by phrases such as 'content knowledge' or 'pedagogical content knowledge'" (p. 19). Silverman and Thompson's (2008) model for mathematical knowledge for teaching, which builds from Thompson and Thompson's (1996) conception, involves powerful understandings and ways of thinking that allow a teacher to act spontaneously in ways that promote conceptual teaching. Conceptually oriented teachers express themselves "in ways that focus students' attention away from thoughtless applications of procedures and toward a rich conception of situations, ideas, and relationships among ideas" (Thompson et al., 1994, p. 7).

In Silverman and Thompson's (2008) view, the development of mathematical knowledge for teaching involves developing significant personal understandings of a particular mathematical topic and transforming these personal understandings to understandings and ways of thinking that are pedagogically powerful. According to Silverman and Thompson (2008), both the personally powerful understandings and mathematical knowledge for teaching develop via a process that Piaget (2001) called reflective abstraction.

Although, the idea of pedagogical content knowledge has been elaborated in numerous studies (e.g., Ball & Bass, 2003; Ball, Bass, Sleep, & Thames, 2005; Ball, Thames, & Phelps, 2008; Carpenter, Fennema, Peterson, & Carey, 1988; Grossman, 1990; Ma, 1999), there has been little clarification of what constitutes it or research into its development. In addition, the majority of studies that have investigated mathematical knowledge for teaching (or pedagogical content knowledge) have been with pre-service teachers at the elementary level (e.g., Ball & Bass. 2003; Hill, Ball, & Schilling, 2008; Marks, 1990; Carpenter, Fennema, Peterson, & Carey, 1988). The current study contributes to filling both of these voids by exploring in-service secondary school teachers' cognitions as they engage in activities designed to promote mathematical knowledge for teaching.

If we assume that the development of mathematical knowledge for teaching, as indicated by Silverman and Thompson (2008), occurs via reflective abstraction, then investigating teachers' cognitions as they reflect on their practice would seem a propitious area of research. Therefore, from a research perspective, the current study will attempt to identify the understandings and ways of thinking that support or constrain teachers' capacity to reflect on their practice. However, reflecting on one's practice entails thinking about more than the mathematics you will teach. It also entails thinking about the mathematical realities of students who will learn that mathematics and the tasks you will use in your teaching. As such, the study will attempt to answer the following research questions:

1) In what ways do teachers' mathematical understandings and ways of thinking support or constrain their capacity to reflect on their practice?

2) In what ways do teachers' images of their students' mathematics support or constrain their capacity to reflect on their practice?
3) In what ways do teachers' images of the tasks they will employ in their teaching, in relation to their mathematical understandings and their understandings of their students' thinking, support or constrain their capacity to reflect on their practice?