# Mathematics Teacher Educator

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Characterizing Secondary Teachers' Structural Reasoning



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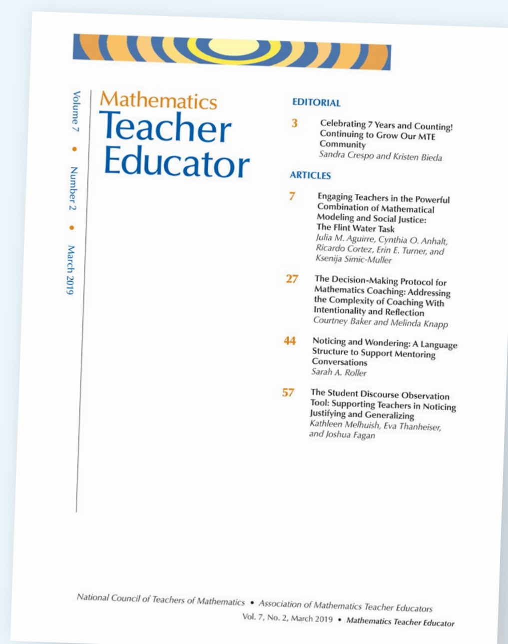
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EDITORIAL



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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

# Characterizing Secondary Teachers' Structural Reasoning

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The Common Core State Standards for Mathematical Practice asks students to look for and make use of structure. Hence, mathematics teacher educators need to prepare teachers to support students' structural reasoning. In this article, we present tasks and rubrics designed and validated to characterize teachers' structural reasoning for the purposes of professional development. Initially, tasks were designed and improved using interviews and small pilot studies. Next, we gave written structure tasks to over 600 teachers in two countries and developed and validated rubrics to categorize responses. Our work contributes to the preparation and support of mathematics teachers as they develop their own structural reasoning and their ability to help students develop structural reasoning.

***Keywords:*** structure sense; substitution principle; mathematical practice

Mathematics teacher educators are tasked with preparing future teachers to engage students in the mathematical practices listed in the Common Core State Standards for Mathematics (CCSSM). One of the more elusive of the Standards for Mathematical Practice is SMP 7: Look for and make use of structure (National Governors Association Center for Best Practices [NGA Center] & Council of Chief State School Officers [CCSSO], 2010). We see this practice in different places throughout students' mathematical careers. In arithmetic, students impose structure implicitly on expressions via order of operations and describe equivalence of expressions via the associative, commutative, and distributive properties. In algebra, students generalize and extend their arithmetic understandings to apply the aforementioned properties in the context of solving equations and understanding functions. In calculus, students apply their structural awareness to determine which rules of differentiation and integration are appropriate for a given function. The entire subject of abstract algebra is dedicated to the study of the underlying structure of mathematical objects. As students transition from each course to the next, they begin to identify relationships among concepts, objects, and techniques. The development of these relationships supports students' awareness of structure (Mason et al., 2009).

As important as looking for and making use of structure is, assessing this practice is challenging for several reasons. In the literature review, we highlight that consensus does not exist about what *structure* means. Moreover, creating items that enable researchers and math teacher educators to distinguish between when a person is *making use of structure* or carrying out a memorized algorithm based on visual or contextual cues is nontrivial. In this article, we share tasks and corresponding rubrics designed to create opportunities for teachers to engage in behaviors linked to SMP 7. Our aim is twofold. First, we want to disseminate these items and rubrics so our fellow math teacher educators have tools to gain insight into how their preservice teachers and secondary teachers may be reasoning about structure. Second, we want to share how math teacher educators can use these rubrics to help organize teacher responses for instructional purposes. Empowered with the knowledge generated by using both the items and rubrics, we believe math teacher educators can more thoroughly engage their preservice and in-service teachers in SMP 7.

## Literature Review

CCSSM does not contain a singular clear definition for *structure, structural reasoning, reasoning about structure*, or any other variation of this phrasing. People use these phrases and generally seem to have an operational understanding of what they mean but are limited to providing illustrative examples rather than precise definitions. We experience the same struggle.

Early work focused on describing structural reasoning strictly in the context of the learning and teaching of algebra. Much of the work in the 1980s and early 1990s focused on structural reasoning as an ability to identify equivalent forms of an expression (Kieran, 1988) and an ability to select appropriate forms for a given task (Linchevski & Vinner, 1990). Kirshner (1989) cautioned that some students attend to surface features and visual cues when engaging in algebraic tasks, rather than attending to the algebraic relationships represented. Linchevski and Livneh (1999) coined the term *structure sense* to signify the use of arithmetic structures in the transition to algebra. Hoch (2003) gave a broad definition of structure sense as "an ability to recognize algebraic structure and to use the appropriate features of that structure in the given context as a guide for choosing which operations to perform" (p. 2). Hoch and Dreyfus (2006) defined specific abilities related to structure sense in various contexts to make Hoch's (2003) general definition useful for guiding student learning and curriculum design.

The variety in this early work captures the fact that the notion of structure is broad and has proven elusive to define. In 2009, Mason et al. defined *mathematical structure* as "the identification of general properties which are instantiated in particular situations as relationships between elements" (p. 10) and *structural thinking* as "a disposition to use, explicate, and connect these properties in one's mathematical thinking" (pp. 10–11). Harel and Soto (2017) adopted the American Heritage Dictionary's definition of structure as "something made up of a number of parts that are held or put together in a particular way" (p. 226), where "in mathematics the way these 'parts' are held together is not restricted to physical or mental spatial configurations (p. 226)." They then define *structural reasoning* as

> **a combined ability to: (a) look for structures, (b), recognize structures, (c) probe into structures, (d) act upon structures . . . (e) reasoning in terms of general structures . . . (f) the ability to see (be aware of) how a piece of knowledge acquired resolves a perturbation experienced. (Harel & Soto, 2017, p. 226)**

In addition to SMP 7 ("look for and make use of structure"), the first high school algebra content standards include HSA.SSE.A.1b, "interpret complicated expressions by viewing one or more of their parts as a single entity," and HSA.SSE.A.2, "use the structure of an expression to identify ways to rewrite it" (NGA Center & CCSSO, 2010). Note that these standards point to the need for students *to identify* structure—meaning a student must be aware that structure is something to look for in representations of mathematical objects—and the need for students *to act* accordingly. Multiple studies have shown a preponderance of weak structure sense among students (Hoch, 2003; Hoch & Dreyfus, 2006; Linchevski & Livneh, 1999; Novotná & Hoch, 2008; Novotná et al., 2006; Tall & Thomas, 1991). This highlights a need to support teachers in helping their students develop structure sense.

One component of structure sense, known as the substitution principle, states that "if a variable or parameter is replaced by a compound term (product or sum), or if a compound term is replaced by a parameter, the structure remains the same" (Novatná & Hoch, 2008, p. 95). The substitution principle is described as "chunking" by Cuoco et al. (2010, p. 686), and Hawthorne and Druken (2019) give examples of this principle in the context of secondary mathematics. They describe "decomposing (or chunking) algebraic expressions into a variety of substructures based on the context and goals at hand" (p. 298) and give examples of how chunking can be useful in solving equations, as well as finding the domain and range of a function. This component of structural reasoning is the focus of our article.

## Project Background

A group of mathematicians, mathematics educators, statisticians, psychometricians, and secondary mathematics teachers formed Project Aspire and created the assessment named *Mathematical Meanings for Teaching secondary mathematics* (*MMTsm*). The *MMTsm* includes items about a wide variety of secondary topics, including function, function notation, rate of change, proportionality, frames of reference, measurement, covariation, and structure sense (Thompson, 2015).

The *MMTsm*, a 46-item written diagnostic instrument, was created with practical and research goals in mind (Byerley & Thompson, 2017; Thompson, 2015; Thompson et al., 2017; Yoon & Thompson, 2020). The *MMTsm* was designed to help professional development leaders diagnose their teachers' meanings for mathematical ideas for planning workshops and to determine if their workshops resulted in teachers developing meanings that are productive in a wide variety of mathematical circumstances. The complete *MMTsm* is freely available in English by visiting https://tinyurl.com/MMTsmDocuments, and a Korean version is available upon request. For research purposes, the *MMTsm* provides a way to collect large-scale data about secondary teachers' mathematical meanings that have previously been investigated only in small, qualitative studies. For practical purposes, the instrument is not meant for evaluation or promotion of in-service teachers nor for grading of preservice teachers; rather, the instrument can provide useful insight for professional development and course planning.

## Item Design

In designing the structure sense items and rubrics discussed in this article, we wanted to know how teachers respond to a given equation or expression for which using the substitution principle would be productive for problem solvers. For example, recognizing the expression $(4x^2 + 3)^2 - (x + 1)^2$ as the difference of two squares (i.e., positive values) allows a problem solver to reason that the graph of $y = (4x^2 + 3)^2 - (x + 1)^2$ has a positive range without having to expand the expression, build a table, or rely on a calculator because $4x^2 + 3 > x + 1$ for all $x$. A teacher's written response to our tasks can indicate that a teacher did or did not use the substitution principle to reason about the structure of a particular equation or expression. We grouped responses that we hypothesized would convey similar meanings to hypothetical students, even if their responses did not use the substitution principle. An explicit goal of the project was to develop a tool that could be used on a large scale to model teachers' thinking by making use of models of teachers' thinking developed in small qualitative studies.

The multiyear process of refining and discarding items was based on our theoretical perspective, constructs from existing literature, and our teaching experiences. The project team drafted items around central ideas such as covariation, functions, and structure, engaging in many in-depth discussions about each item. The team then conducted several rounds of task-based clinical interviews (Goldin, 2000) with preservice teachers and secondary teachers. The goals for these interviews included ascertaining if the respondents interpreted items as designed, and validating our characterization of the responses and thinking the items elicited from teachers. After revisions, the team created a paper-and-pen packet that 251 U.S. teachers completed before larger scale use. More details of this process are discussed in Thompson (2015).

## Rubric Design and Validation

We categorized teachers' written responses to items with a grounded-theory approach, using existing research literature to build theoretical sensitivity (Corbin & Strauss, 2014). For each task, we grouped the most common responses into levels. For all tasks, we added a category for "I don't know" (or equivalent) responses, a category for responses left blank, and a Level 0 for all responses that did not fit any other category. We designed the rubrics to be used by people with an undergraduate level of mathematics and a few days of training, so undergraduate students could participate in classifying responses. See the **Appendix** for abbreviated rubrics for the shared items and more detailed rubrics. Versions of the detailed rubrics that include examples at each level with commentary are available by contacting the authors.

After creating draft rubrics, we used the rubrics on new teacher responses, improved the rubrics through discussion, and eventually computed interrater reliability (IRR) scores. For IRR, we made use of both percent agreement and Cohen's $\kappa$, which improved upon percent agreement by adjusting for the agreement by chance (Cohen, 1960). We had lower than desired IRR scores on some items during initial scoring trials. We examined the teachers' responses that had poor scorer agreement and modified the rubrics to help scorers make decisions about borderline responses. In the Tasks and Rubrics section, we list our final IRR scores after many rounds of rubric refinement.

We also administered a Korean version of the *MMTsm* to 366 South Korean teachers. The translation and back-translation processes are described in Yoon et al. (2015). Yoon trained South Korean mathematics education graduate students to use the rubrics to score South Korean teachers' responses. We computed IRR scores by comparing Yoon's scores to those of the South Korean scorers. Using the *MMTsm* and rubrics in South Korea helped us improve the clarity of rubrics for other users and provided evidence that the instrument could be used outside of the United States.

# Tasks and Rubrics

We present three tasks, one grounded in the context of expressions and two in the context of equations. For each task, we provide a brief description of the task, a rubric for how we interpret responses as they relate to the substitution principle and structural reasoning, and select teacher responses.

## Task 1: Associative Property

Our first task centers on teachers' reasoning with subexpressions as both whole units and as objects that can be broken apart into smaller units (i.e., chunking and unchunking). Figure 1 shows the prompt we provided to teachers.

### Connection to Substitution Principle

We see at least three ways someone can demonstrate varying degrees of structure sense while engaging with this task. Suppose that an individual sees the expression as entailing four distinct entities: $u$, $v$, $w$, and $z$. If the person does not engage in chunking, they might respond to the item with a statement that "it isn't possible to apply the property because there are four items, not three." Suppose a second individual looks at the expression and sees two quantities: $(u \Delta v)$ and $(w \Delta z)$. This individual has applied the substitution principle by chunking both pairs instead of just one pair (see Figure 2). They might now respond, “No, the property can’t be applied because you need one more number.” Finally, consider a third individual who engages in both chunking and unchunking; that is, they use the substitution principle flexibly. This individual would respond that the property does apply.

Now consider a teacher’s response to this task. If the teacher is aiming to support their students in reasoning with the substitution principle, they will break apart the given expression into three objects, one containing two elements (as in ($w \Delta z$)) and two others as single elements, $u$ and $v$, separated by the operator (see Figure 3).

**Rubric and Select Responses**

Table 1 shows an abbreviated version of the rubric for the Associative Property task. In terms of IRR, we reached 89% agreement and a Cohen’s κ of 0.86 when applying the rubric to responses from U.S. teachers; we reached 80% agreement and a Cohen’s κ of 0.71 when scoring South Korean teachers’ responses.

**Figure 1**

*Associative Property Task*

Δ is an operation with the following property.

For all real numbers, $a$, $b$, and $c$, $(a \Delta b) \Delta c = a \Delta (b \Delta c)$.

Let $u$, $v$, $w$, and $z$ be real numbers. Can this property of Δ be applied to the expression below? If yes, demonstrate as if to students. If no, explain to students why it cannot.

$$(u \Delta v) \Delta (w \Delta z)$$



**Figure 2**

*Thinking of the Expression as Two or Four Objects*

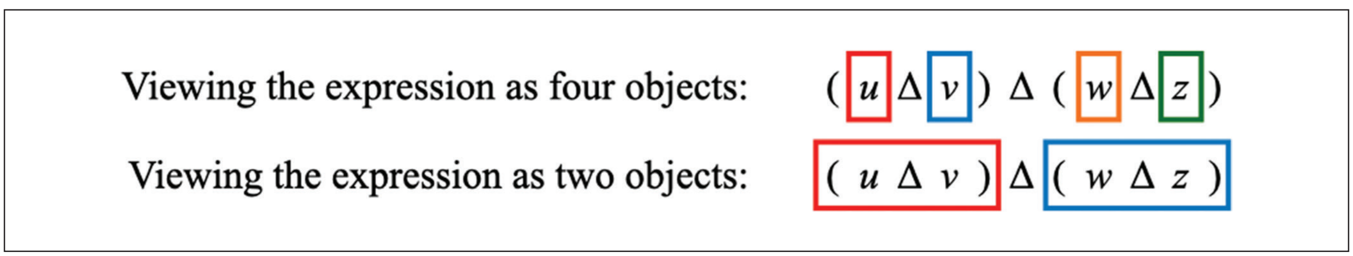
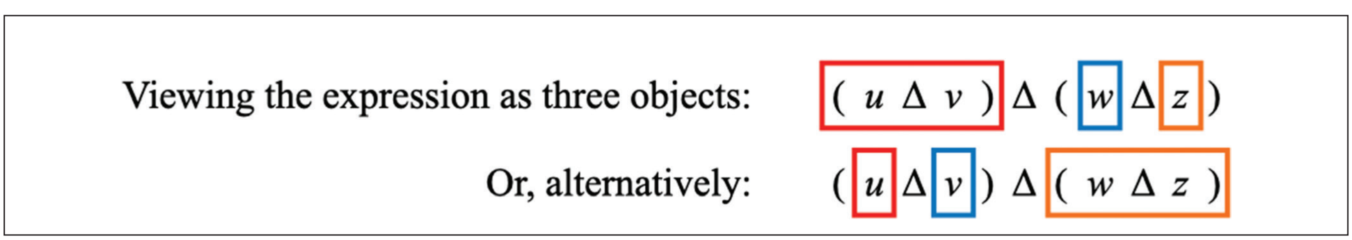



**Figure 3**

*Using the Substitution Principle Flexibly to Identify Three Objects*



The distinction between a Level 4 and a Level 3 response reflects the task's prompt requesting responses as if explaining to a student. A Level 4 response has an explicit reference to the chunking of objects in one set of parentheses. In a Level 3 response, no such chunking or substitution is made explicit, but the response is still correct.

We hypothesize teachers who provided Level 2 responses focused on the changing position of parentheses in the statement of the property, rather than the prescribed regrouping of terms (see Figure 4). In such instances, the teacher would not be engaging in reasoning with the substitution principle. Rather, their reasoning might be centered on a seemingly arbitrary rearrangement of parentheses. Although moving parentheses arbitrarily is a great strategy for doing mental arithmetic, attending to structure is necessary when applying definitions and properties in courses such as geometry in high school and proof-based courses at the university.

Interviews with teachers provide evidence supporting our earlier description of how a teacher might conclude the property does not apply (Level 1 response). Figures 5 and 6 exemplify Level 1 responses. Figure 5, drawn from an interview, shows the response of a teacher who chunked the elements into the two groups ($u \Delta v$) and ($w \Delta z$), leading them to declare the need for another element. Figure 6 shows work provided by a different teacher claiming the given expression has too many elements.

**Table 1**

*Rubric for Associative Property Task (Level 0, I Don't Know, and Blank Categories Omitted)*

| Level | Level description |
|---|---|
| 4 | All of the following. The teacher—<br>• identified ($w \Delta z$) or ($u \Delta v$) as one object; and<br>• applied the property to provide the expression "$u \Delta (v \Delta (w \Delta z))$" or "$((u \Delta v) \Delta w) \Delta z$" or an equivalent. |
| 3 | All of the following. The teacher—<br>• wrote only "$u \Delta (v \Delta (w \Delta z))$" or "$((u \Delta v) \Delta w) \Delta z$" or an equivalent; and<br>• gave no indication of how they grouped either ($w \Delta z$) or ($u \Delta v$) into one element. |
| 2 | The response is consistent with a meaning for associative property that one can move or omit parentheses arbitrarily. The teacher used $\Delta$ and the given variable names in their answer. Frequently the teacher wrote "$u \Delta (v \Delta w) \Delta z$" in this category. |
| 1 | The teacher said that the property is not applicable to the given expression. |



**Figure 4**

*Example of Response in Category "Moved Parentheses Arbitrarily"*

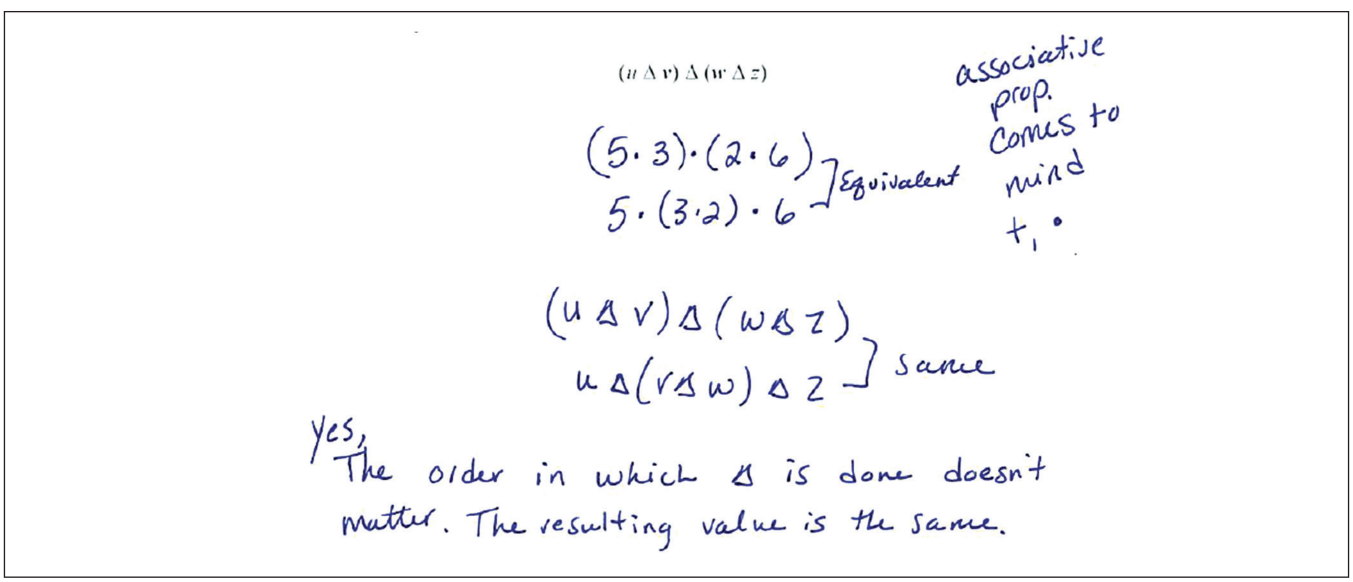
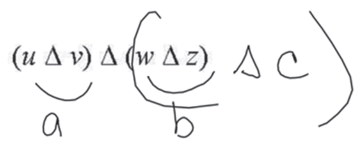


**Figure 5**

*Interview Excerpt With Teacher on the Associative Property Task*

*Teacher:* [Reads problem and then long pause.]
Hmm. Well, I'm thinking to myself, if associative is essentially just a grouping property, and if I called this $a$ [circles $(u \Delta v)$] and that $b$ [circles $(w \Delta z)$] part of me is like, no, I can't associate the groupings because I would need like one more thing.



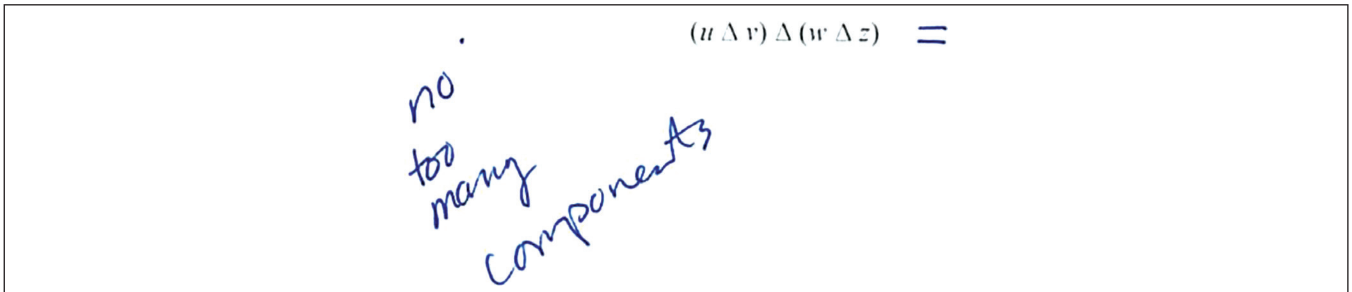

If it were written like that and I snuck in one more delta, then now I could be like sure, associate it like that, there now the associative property just held. I feel like without a third quantity to associative with…I don't know.

*Interviewer:* So your first reaction is to say no?

*Teacher:* Yeah

**Figure 6**

*A Common Justification for Why the Property Does Not Apply*




## Task 2: Solution From Identical Structure

Our second task allows us to determine whether teachers recognize and use structural equivalency to find a solution to the given equation (see Figure 7).

### Connection to Substitution Principle

Of the three tasks, solving this task structurally involves a more classical application of the substitution principle. This item's purpose is to determine to what extent an individual identifies the identical structure between the given equations and uses that structure to solve the task. A productive way of reasoning structurally about this task is to recognize that the equation $3x^5 - 2x^2 + 4 = 0$ has the same structure with respect to its input $x$ as $3(x + 1)^5 - 2(x + 1)^2 + 4 = 0$ does with respect to the argument $x + 1$. In other words, if a person were to substitute $x + 1$ for $x$ in the first equation, they would obtain the second equation. A person may continue to reason structurally in several ways. One person might reason that because $x = -0.942$ will make the first equation true, then $x + 1 = -0.942$ must make the second equation true. Another individual might recognize that the graph of $y = 3(x + 1)^5 - 2(x + 1)^2 + 4$ is the graph of $y = 3x^5 - 2x^2 + 4$ shifted horizontally; then, they reason about graph transformations to determine the new $x$-intercept.

As Hawthorne and Druken (2019) noted, individuals can look at the same mathematical object and see different structures. For example, an individual could focus on the binomial expressions in the first two terms on the left-hand side of $3(x + 1)^5 - 2(x + 1)^2 + 4 = 0$. This individual might pursue expanding the binomials and collecting like terms. While this individual could continue on to attempt to solve algebraically or graph the resulting polynomial, they are not capitalizing on all the structural information provided in the task. By making the substitution principle a core component of their structural reasoning, an individual can make strategic choices in their problem solving.

**Rubric and Sample Responses**

Table 2 shows the abbreviated rubric for the Solution From Identical Structure task. For IRR, we reached 92% agreement and a Cohen's κ of 0.88 when applying the rubric to responses from U.S. teachers; we reached 100% agreement and a Cohen's κ of 1 when scoring South Korean teachers' responses.

Levels 3a and 3b distinguish between teachers whose first attempts showed that they immediately recognized and utilized the identical structure to solve the task and those who first tried another method(s) before leveraging the identical structure. We place both types of responses at Level 3 because they reflect utilizing the substitution principle. The left of Figure 8 shows two teachers' complete responses at Level 3a. On the right of Figure 8, we show the work of a teacher who began factoring $3(x + 1)^5 - 2(x + 1)^2 + 4$ before stopping and making explicit use of the substitution

**Figure 7**

*Solution From Identical Structure Task*

The equation $3x^5 - 2x^2 + 4 = 0$ has $x = -0.942$ as a solution. What is a solution to $3(x+1)^5 - 2(x+1)^2 + 4 = 0$?



**Table 2**

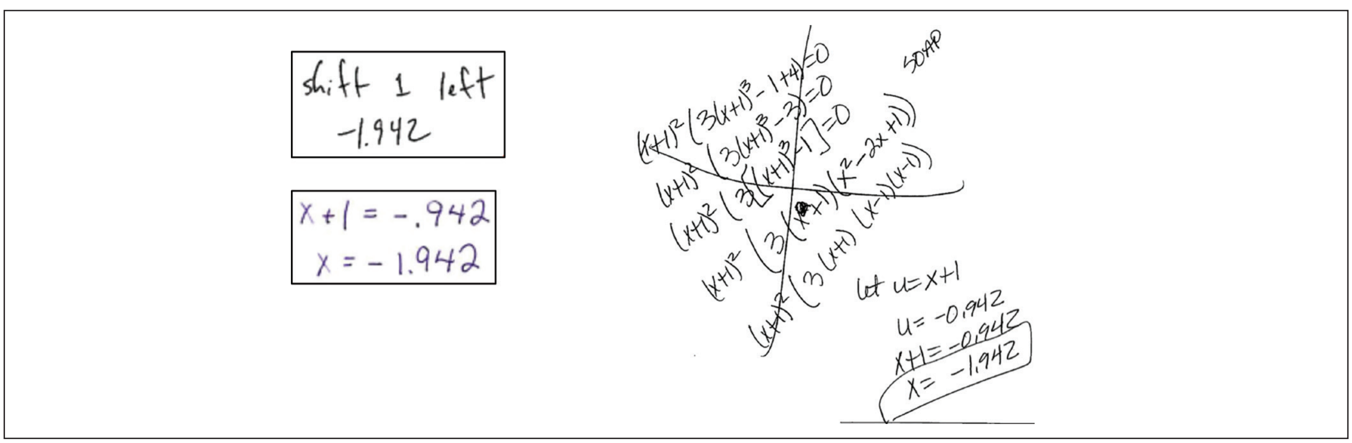
*Rubric for Solution From Identical Structure Task (Level 0, I Don't Know, and Blank Categories Omitted)*

| Level | Sublevel | Level description |
|---|---|---|
| 3 | a | The teacher found the solution $x = -1.942$. |
| | b | The teacher concluded that $x = -1.942$ after first trying a different solution or method. |
| 2 | | The teacher found a solution by *adding* 1 to the given value of $x$. |
| 1 | | The teacher wrote that $x = -0.942$ is a solution to the new equation, regardless of explanation. |



**Figure 8**

*Sample Responses for Levels 3a (Left) and 3b (Right) on Solution From Identical Structure*



principle. The split in Level 3 enables math teacher educators to identify opportunities to engender conversations about various structural approaches to problem solving. For instance, the right of Figure 8 helps us envision having a conversation about how capitalizing on one structural element—the binomial—might lead to expanding the expression. Such work might quickly become distasteful to the solver, leading them to abandon the approach in favor of using other structural elements. We will return to the matter of a/b sublevels in the Discussion section.

In Figure 9, we show part of an interview with one teacher whose work we categorized as Level 3a. We trace the teacher's reasoning as they identify the identical roles of $x$ in the first equation and $x + 1$ in the second. The teacher then reasons "this whole thing" (i.e., $x + 1$) must be equal to –.942 and so "the $x$ in $x + 1$" must be –1.942.

The lower levels of the rubric reflect categories we made to capture common responses that were not consistent with the aforementioned ways of structural reasoning. Our intent here is to help provide math teacher educators with useful information about how their in-service and preservice teachers might respond to the task. We have also found that showing sample lower-level rubric responses to teachers in workshops promotes fruitful discussion about the importance of looking for and making use of structure. We focus discussion on the mathematics in the response and usually identify responses as student work rather than teacher work.

## Task 3: $x^4$ on Horizontal Axis

The third task centers on graphing an equation in which the horizontal axis does not represent values of $x$, but rather $x^4$ (Figure 10). We reformatted the axes in the task for ease of presentation here; the respondents were given a larger set of axes on which to graph. On the basis of our data, we suggest that this task can be challenging because solving it involves the substitution principle, a nonstandard horizontal axis, and consideration of domain.

**Figure 9**

*Interview Excerpt of a Teacher Reasoning Through the Solution From Identical Structure Task*

*Teacher:* You know $x$ has to equal that. What is being cubed into the 5 element ….

$$3(-1.942+1)^5 - 2(-1.942+1)^2 + 4 = 0$$

And so we have an equivalence.

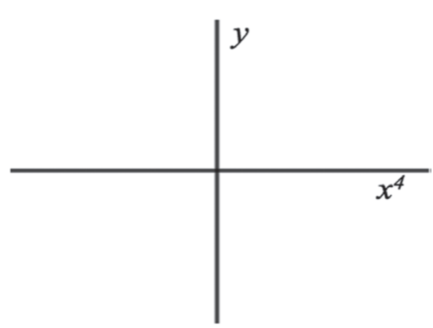
$$x+1 = -.942$$
$$x = -1.942$$

Right [speaking to himself]. But the fact is, what I'm doing here is squaring the input. So, I'm still squaring the input [sic], but we know from the statement we know that the input has to be equal to –0.942, So how do I make this whole thing –0.942. I have to figure out what $x$ has to be in $x + 1$ so I can create that.

**Figure 10**

*$x^4$ on Horizontal Axis Task*

Suppose values of $x^4$ are on the horizontal axis of a coordinate system and $y$ is on the vertical axis. Sketch the graph of $y = x^2 + 5$.



### Connection to Substitution Principle

A productive way of reasoning about this item involves creative use of the substitution principle. The substitution principle shows up in two places. First, the individual treats $x^4$ as $u$, thus envisioning a $uy$-graph. Second, the individual writes $x^2$ in terms of $u$, namely, $x^2 = u^{1/2}$. Thus, the function $y = x^2 +5$ is equivalent to $y = u^{1/2} +5$. This allows the individual to conceptualize the graph of $y = x^2 + 5$ as the graph of $y = u^{1/2} + 5$ when the horizontal axis represents values of $u = x^4$. Because all values of $x^4$ are nonnegative, the graph of the function appears only at and to the right of the $y$-axis.

Of the three tasks we have presented, the $x^4$ on the Horizontal Axis task involves the least standard application of the substitution principle, in part because of its graphing context. This way of reasoning structurally is of particular importance in statistics and data science. When a scatter plot suggests that a curvilinear relationship might exist between the dependent and independent variables, say $y = ax^2 + c$, statisticians and data scientists will use the substitution principle to replace $x^2$ with $w$ to then work with the relationship $y = aw + c$. They might even recreate the scatter plot of $y$ vs. $w$ in addition to using a general linear regression model for $y$ given $w$. Thus, this way of structural reasoning is vital to statisticians and data scientists because it allows them to generalize the ordinary least squares method in powerful ways.

### Rubric and Select Responses

Table 3 shows an abbreviated rubric for the $x^4$ on Horizontal Axis task. The full rubric has graphs and sample responses. In terms of IRR, we reached 88% agreement and a Cohen's κ of 0.74 when applying the rubric to responses from U.S. teachers; we reached 100% agreement and a Cohen's κ of 1 when looking at South Korean teachers' responses.

Within the rubric for this task, we make use of sublevels to help distinguish between different cases. However, we do not consider there to be a hierarchy between the sublevels. For example, we do not view Level 4a to be "higher" or "better" than Level 4b or vice versa. However, we do view both Levels 4a and 4b as being more productive than Levels 3a and 3b with respect to implications for student learning.

Consider the work shown in Figure 11. On the left, the teacher explicitly used the substitution principle, while on the right, a second teacher made use of a table to help build their graph. We created two rubric categories in Level 4 to capture the fact that both teachers produced correct graphs, although their underlying reasoning might have differed.

We believe solving the problem with a table differs from solely using the substitution principle because using tables has some constraints. In our experience, individuals might plot a few points using a table and then make erroneous assumptions about the rest of the graph. The response in Figure 12 shows such an instance. The teacher made a table containing positive values of $x$ and corresponding values of $y = x^2 + 5$. The teacher might have remembered that even functions are symmetric about the $y$-axis (with $x$ on the horizontal axis) and reflected the function's graph over the $y$-axes without considering that values of $x^4$ are always positive to construct the graph shown.

Figure 13 shows an example of teacher who used the substitution principle to rewrite $x^4$ in terms of $u$. The teacher then noted that $x^2 = \pm\sqrt{u}$, which contains the symmetry matching the sideways parabola they sketched. We included this as an example of how the substitution principle can be applied in a problem-solving strategy without necessarily leading to a correct solution.

**Table 3**

*Rubric for $x^4$ on Horizontal Axis (Level 0, I Don't Know, and Blank Categories Omitted)*

| Level | Sublevel | Level description |
|---|---|---|
| 4 | a | The teacher's graph has the same shape as the graph of $y = \sqrt{x} + 5$ (graphed on the standard $xy$-plane), with domain $[0,\infty)$ and $y$-intercept (0,5), and response contains no work showing the use of a table. |
| | b | The teacher's graph has the same shape as the graph of $y = \sqrt{x} + 5$ (graphed on the standard $xy$-plane), with domain $[0,\infty)$ and $y$-intercept (0,5), and response contains work showing the use of a table. |
| 3 | a | The teacher's graph has the same shape as a root function (such as $y = \sqrt{\lvert x\rvert} + 5)$ ) graphed on the standard $xy$-plane with domain $(-\infty,\infty)$ and $y$-intercept (0,5). |
| | b | The teacher's graph has the same shape as the graph of $x = (y - 5)^2$ (graphed on the standard $xy$-plane) with domain $[0,\infty)$ and vertex (0,5). |
| 2 | | The teacher's graph is an upward-facing parabola with domain $(-\infty,\infty)$ or $[0, \infty)$ and vertex (0,5). |
| 1 | | The teacher graphed a collection of at least three points without connecting them. |

**Figure 11**

*Sample Responses for Levels 4a (Left) and 4b (Right) on the $x^4$ on Horizontal Axis Task*

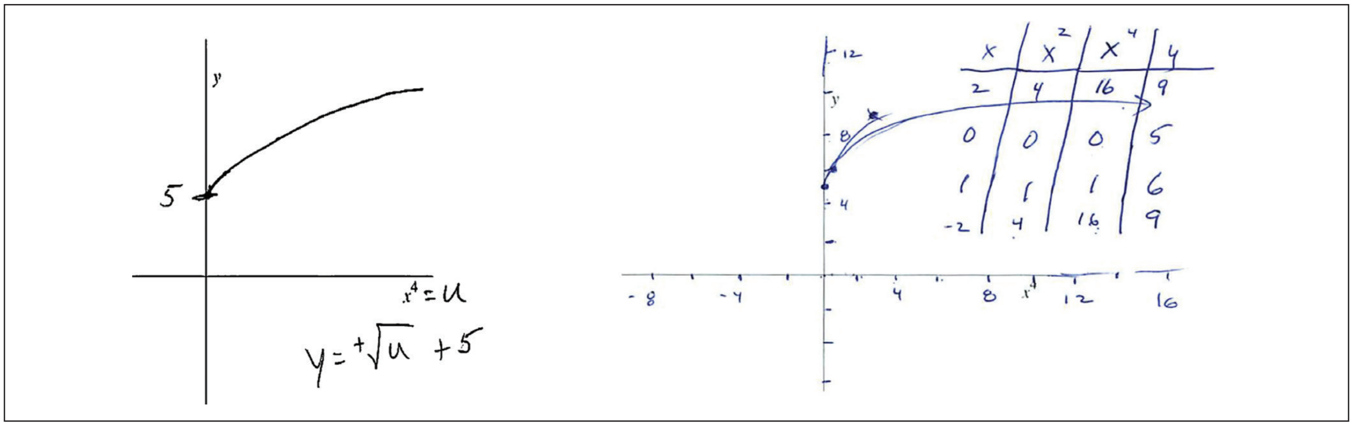



**Figure 12**

*A Level 3a Response to the $x^4$ on Horizontal Axis Task*

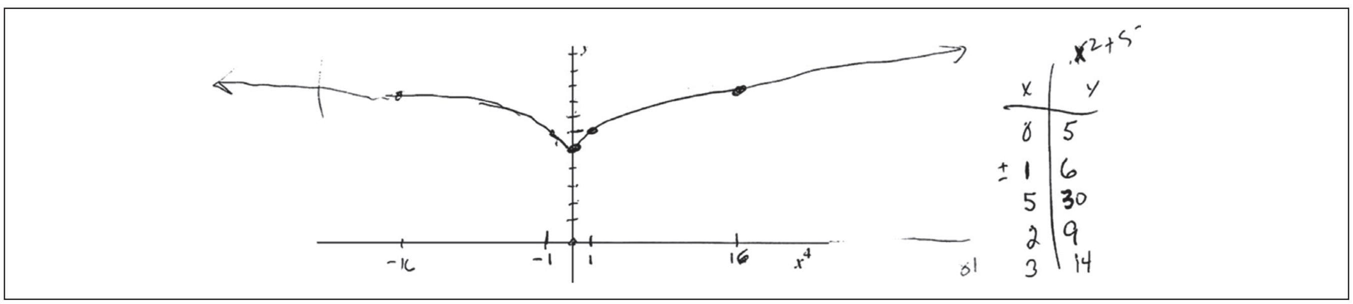
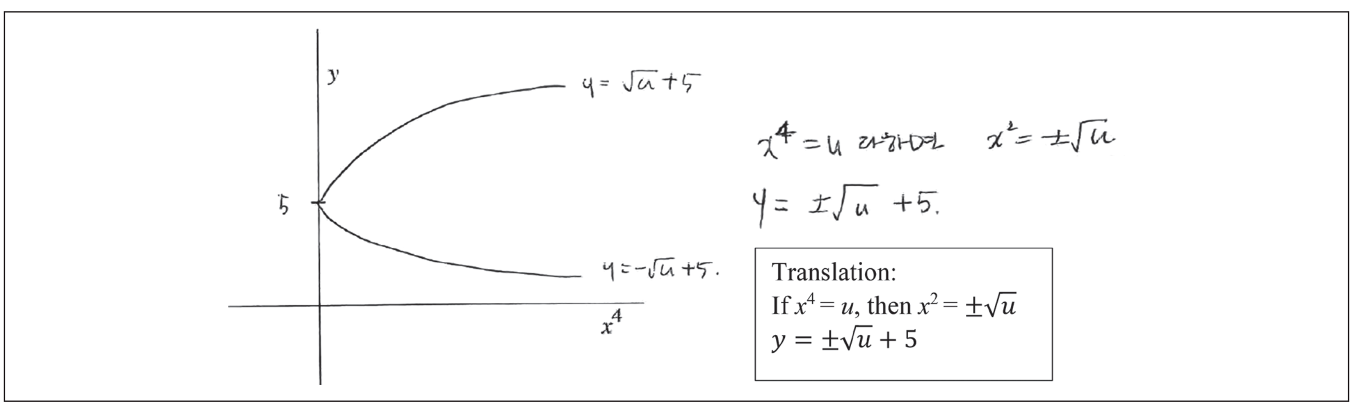



**Figure 13**

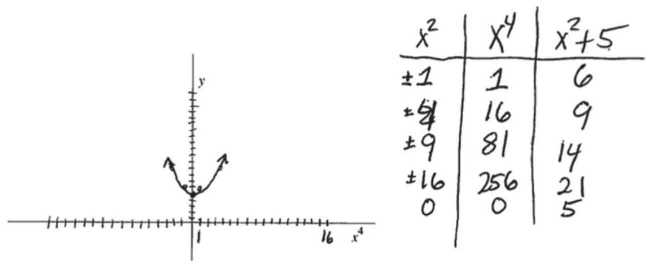
*A Level 3b Response to the $x^4$ on Horizontal Axis Task*




We present Figure 14 as an example of a Level 2 response. Rather than drawing on the substitution principle or making use of symmetry, we believe the teacher viewed $x^4$ as a transformation of $x$ that yields the first coordinate of each point and the equation $y = x^2 + 5$ gives the second coordinate. The last two columns of the teacher's table give the ordered pairs they then plotted. To connect the points, we suspect that the teacher recognized the quadratic structure

**Figure 14**

*A Level 2 Response to the $x^4$ on Horizontal Axis Task*




of $y = x^2 + 5$ to create a parabolic curve. As such, one could argue that this teacher saw and made use of structure (i.e., quadratic equations yield parabolic curves), but did so without coordinating all information given in the task.

## Discussion

The *MMTsm* consists of tasks meant to provide insight into how respondents might be thinking and rubrics that assist in organizing their responses meaningfully to frame professional development. Mathematics teacher educators have been surprised by their teachers' responses on the structure items and have noticed that examining the teachers' work helped them identify gaps in their professional development programs. Some assumed that the structure tasks would be easy for teachers with mathematics degrees, but mathematics teacher educators found that these tasks challenged many successful mathematics teachers. We hypothesize this is not due to a limitation in the teachers' potential to look for and make use of structure, but more of a limitation in their academic preparation. On the basis of our experience hosting a workshop, we observed teachers make progress on using structural reasoning in many contexts once they worked with multiple items where it was advantageous.

As a further example of using these items and rubrics, the second author administered the complete 46-item *MMTsm* to her preservice teachers. From the results, she assigned each preservice teacher to construct and teach a lesson on a specific item they personally did well on to the rest of the class. The rubrics provided preservice teachers with examples of alternative ways of thinking to address in their lesson. Giving preservice teachers a rubric helped them anticipate their peers' thinking and categorize responses when they led class discussions on their task (Smith & Stein, 2018). In this manner, the levels within the rubrics provided support for mathematics teacher educators to help preservice teachers develop awareness of how other people might reason about mathematical tasks.

We find that the a/b sublevels can further support nuanced discussions about the sometimes-subtle distinctions in responses that otherwise convey similar results. In the Solution From Identical Structure task, responses at Levels 3a and 3b are both correct; however, Level 3b separates out responses that contain shifts in approaches from those that do not. Having these responses sifted out gives mathematics teacher educators the opportunity to draw sample responses for discussion about problem-solving strategies. As an example, Figure 15 shows a teacher creating a similar but simpler problem to solve first. This allowed the teacher to build a way of reasoning they could transfer to the original problem.

In the $x^4$ on Horizontal Axis task, Level 4a allows for a conversation centering on using the substitution principle in conjunction with recognizing and making use of images of parent graphs. Level 4b highlights that we have a tool (making a table) to determine points on a graph, which could lead one to recognize the square-root relationship between the outputs and values on the horizontal axis. The distinctions in Levels 3a and 3b necessitate a discussion about the underlying domain of the function.

Our items also provide varied contexts in which math teacher educators may create opportunities for their teachers to engage with SMP 7. Although some teachers consistently used the substitution principle across the items, noticeably more teachers gave responses that varied greatly in terms of their application of the substitution principle. Only 27% of U.S. teachers who used the substitution principle (a Level 3 or 4 response) on the Associative Property task also did so on the $x^4$ on Horizontal Axis task (a Level 4a response). Similarly, 12% of the U.S. teachers who drew on substitution for the Solution From Identical Structure task (Levels 3a or 3b) continued to do so for the $x^4$ on Horizontal Axis task. Thus, using the substitution principle profitably in one instance does not imply a teacher will use it in another. These results stem from a convenience sample (described in Byerley & Thompson, 2017), so we caution the reader to not generalize these statements to all U.S. teachers.

Considering the large variation in individuals' usage of the substitution principle across our tasks, we propose that mathematics teacher educators need to use a variety of contexts, moving beyond algebraic expressions and equations, to work on teachers' structural reasoning. By modeling SMP 7 in many and varied contexts, preservice and in-service teachers will have a wider selection to use with their own students. We note that the *MMTsm* has five additional items dealing with structure beyond the substitution principle that we have not presented here.

In addition to talking about structural reasoning in various contexts, we propose that mathematics teacher educators distinguish between structural reasoning as a polished outcome and the reality that an individual might see and act upon a structure that does not automatically lead to a solution. We noticed that the literature refers to structural reasoning and structure sense as if such thinking guarantees a normatively correct solution. Hawthorne and Druken (2019) describe steps in helping students develop a *structural lens*, with the last step requiring students "to pause to examine the structure and decide whether one manipulation may simplify a problem more than another" (p. 300). The implication is that students will choose the "right" structure and manipulation. However, we saw responses like that in Figure 16, in which a teacher commented on structural elements and yet did not capitalize on them. Such responses signal the need for explicit conversations with teachers about varied structures and how to leverage them when problem solving. When using these tasks with teachers, we feel it is acceptable to categorize responses as more or less productive for student learning, but it is critical to focus categorization efforts on particular responses—not the teachers' overall mathematical ability.

The *MMTsm* and associated rubrics can provide valuable information for mathematics teacher educators and serve as objects of discussion and reflection with teachers. In particular, using these tasks and rubrics creates opportunities for mathematics teacher educators to engage teachers in discussing mathematical solutions to problems that, though difficult, could be taught in secondary schools. In our experience, teachers were genuinely interested in discussing different sample responses from rubrics and

**Figure 15**

*Creating a Similar but Simpler Problem*

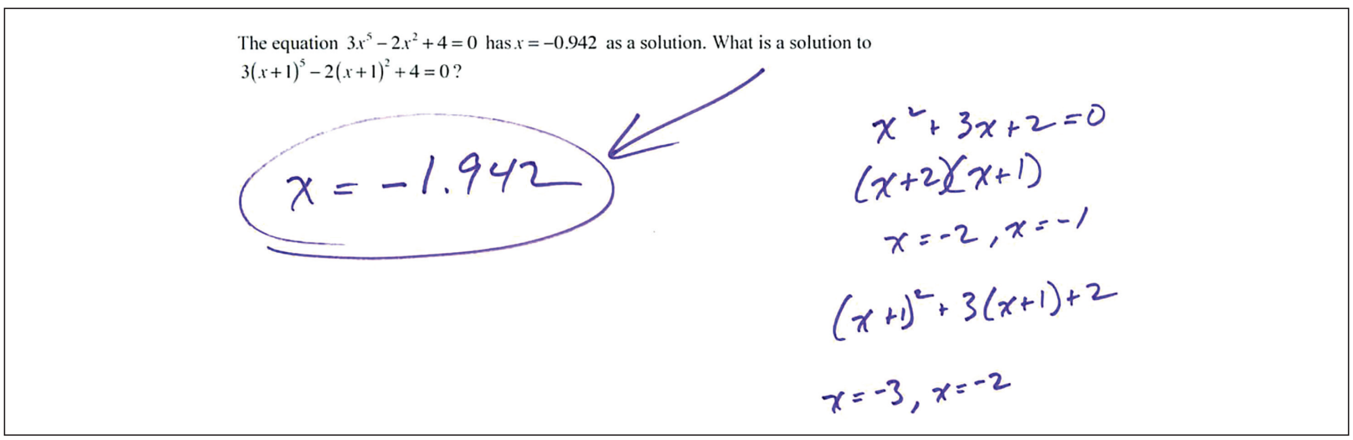
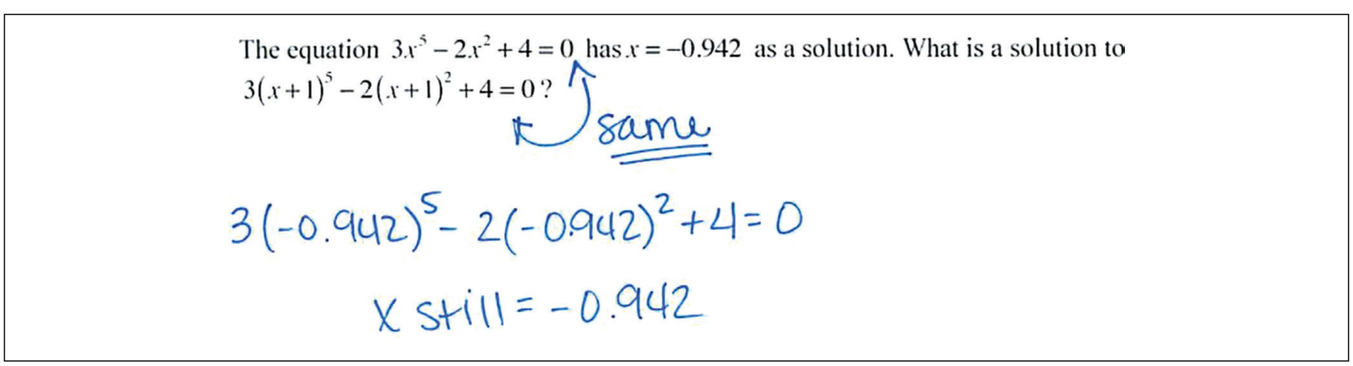



**Figure 16**

*Recognizing Structure but Not Capitalizing on It*



making inferences about what the person who provided that response might have been thinking.

## Acknowledgments

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## Author Note

We have no known conflict of interest to disclose. The entire *MMTsm* is accessible at: https://tinyurl.com/MMTsmDocuments.

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# Appendix: MMTsm Rubrics

Full rubrics with multiple sample teacher responses and additional scoring tips are available by contacting the authors.

***Task 1: Associative Property***

| |
|---|
| Δ is a closed operation on the real numbers with the following property.<br>For all real numbers $a$, $b$, and $c$, $(a \Delta b) \Delta c = a \Delta (b \Delta c)$.<br>Let $u$, $v$, $w$, and $z$ be real numbers. Can this property of Δ be applied to the expression below? If yes, demonstrate as if to students. If no, explain to students why it cannot.<br>$(u \Delta v) \Delta (w \Delta z)$ |

**Purpose**

This item aims to see whether teachers can selectively unitize an expression, treating it as if it is one object even though perceptually it is made of many objects.

**Rationale**

This item is designed to see whether teachers can see one of $(w \Delta z)$ or $(u \Delta v)$ as one object and simultaneously see the other as composed of two objects, and convey this way of thinking to students.

Levels 4, 3, and 2 are designed to capture responses that both selectively unitize and convey this to students, selectively unitize but do not convey this, and do neither, respectively. Level 1 is designed to capture responses that cannot give us any information about selective unitization because of the teacher's meaning for the associative property.

**Scoring Instructions**

- Ignore crossed-out work on this rubric.
- If the teacher's response contains elements that fit into more than one level, assign the lower level score.
- A Level 4, 3, 2, or 1 response that also contains work using numbers and/or addition or multiplication is scored at that level.
- Score a response that **only** uses numbers or **only** interprets Δ as addition or multiplication at Level 0.

**Summary of Levels for the Associative Property Task**

| | |
|---|---|
| **Level 4 response** | ***All* of the following. The teacher:**<br>+ identified (w Δ z) or (u Δ v) as one object<br>+ applied the property to provide the expression "u Δ (v Δ (w Δ z))" or "((u Δ v) Δ w) Δ z" or an equivalent. |
| **Level 3 response** | ***All*** of the following. The teacher:<br>+ wrote only "$u \Delta (v \Delta (w \Delta z))$" or "$((u \Delta v) \Delta w) \Delta z$" or an equivalent<br>+ gave no indication of how he or she grouped either $(w \Delta z)$ or $(u \Delta v)$ into one object. |
| **Level 2 response** | The response is consistent with a meaning for associative property that one can move or omit parentheses arbitrarily. |
| **Level 1 response** | The teacher said that the property is not applicable to the given expression. |

| | |
|---|---|
| **Level 0 response:** | *Any* of the following:<br>– The response does not fit any of the higher levels.<br>– The scorer cannot interpret the response.<br>– The response consists of scratch work with no clear indication of a final answer.<br>– The response does not address the prompt; that is, the response is off-topic (see Purpose and Rationale).<br>– The page contains no work, but does contain at least one mark to suggest that the teacher saw this item. |
| **IDK response** | The response consists only of "I don't know," or something equivalent that suggests that the teacher is unsure of how to respond. If the teacher stated uncertainty and gave an additional response, score the response ignoring the uncertainty. |
| **X response** | The page is completely blank. |

### *Task 2: Solutions From Identical Structure*

The equation $3x^5 - 2x^2 + 4 = 0$ has $x = -0.942$ as a solution. What is a solution to $3(x+1)^5 - 2(x+1)^2 + 4 = 0$?

#### Purpose

This item aims to see if teachers recognize structural equivalency and can use it to find a solution to the given equation.

#### Rationale

This item is designed to see if teachers recognize the structural equivalence of the two equations and can use that to find a solution. If the teacher used a graphing calculator, or otherwise solved for solutions, we hypothesize that that teacher did not recognize the structural equivalence (or did not know how to use it). While we do attend to whether or not the teacher's first instinct is a strategy consistent with getting $x = -1.942$ (Level 3a versus Level 3b), the item does not ask the teacher to explain his or her reasoning. Thus, we do not distinguish between the various ways in which a teacher arrived at the correct answer of $x = -1.942$.

#### Scoring Instructions

- Attend to crossed-out work or scratch work when deciding between Level 3a and Level 3b.
- Ignore crossed-out and scratch work for Levels 2 and lower.
- If a response fits more than one level, score at lowest level.

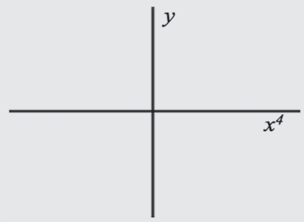
**Summary Levels for the Solutions From Identical Structure Task**

| | |
|---|---|
| **Level 3a response** | The teacher found the solution $x = -1.942$. |
| **Level 3b response** | The teacher concluded that $x = -1.942$ after first trying a different solution or method. |
| **Level 2 response** | The teacher found a solution by *adding* 1 to the given value of $x$. |
| **Level 1 response** | The teacher wrote that $x = -0.942$ is a solution to the new equation, regardless of explanation. |
| **Level 0 response** | **Any** of the following:<br>– The response does not fit a higher level.<br>– The scorer cannot interpret the response.<br>– The response consists of scratch work with no clear indication of a final answer.<br>– The response does not address the prompt; that is, the response is off-topic (see Purpose and Rationale).<br>– The page contains no work, but does contain at least one mark to suggest that the teacher saw this item |

| | |
|---|---|
| **IDK response** | The response consists only of "I don't know", or something equivalent that suggests that the teacher is unsure of how to respond. If the teacher stated uncertainty and gave an additional response, score the response ignoring the uncertainty. |
| **X response** | The page is completely blank. |

***Task 3: $x^4$ on Horizontal Axis***

Suppose values of $x^4$ are on the horizontal axis of a coordinate system and $y$ is on the vertical axis. Sketch the graph of $y = x^2 + 5$.




### Purpose

This item aims to determine whether teachers can think in terms of functions' arguments as opposed to functions' inputs. In this item, think of $x^4$ as if it were $u$. Then $y = u^{1/2} + 5$, so the graph will look like the graph of a square root function with standard labels on the axes (i.e., $x$ on the horizontal axis and $y$ on the vertical axis).

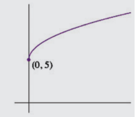
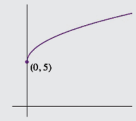
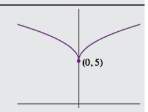
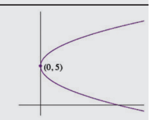
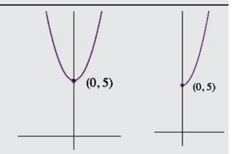
### Rationale

We hypothesize that teachers who can see and make use of structure will be able to reformulate $x^2 + 5$ in terms of $x^4$, so that $y$ is a function of $x^4$. Other teachers might reason about the even power (either $x^2$ or $x^4$) and conclude the graph ought to be symmetric about the $y$ axis. We expect many teachers not to reason structurally, and instead opt for a "plotting points" strategy.

### Scoring Instructions

- Ignore crossed-out work.
- Consider scratch work only to distinguish Level 4a and Level 4b responses by looking to see if the teacher made a table (or computed values as if to fill in a table). Do not consider the validity of the values in the table.
- Focus solely on the sketch itself and do not focus on scaling of horizontal axis.
- Be somewhat lenient when scoring sketches—we do not want to score lower because of artistic ability (see first example in Level 4b).
- If a teacher creates more than one sketch (and/or redraws axes labeled $x^4$ and $y$) matching different level descriptions, score at the lowest level.
- If a teacher creates a new set of axes that are not labeled $x^4$ and $y$, score at Level 0.

**Summary of Levels for the $x^4$ on Horizontal Axis Task**

| | | |
|---|---|---|
| **Level 4a response** | The teacher's graph has the same shape as the graph of $y = \sqrt{x} + 5$ (graphed on the standard $xy$-plane), with domain $[0,\infty)$ and $y$-intercept (0,5), and response contains no work showing the use of a table. |   |
| **Level 4b response** | The teacher's graph has the same shape as the graph of $y = \sqrt{x} + 5$ (graphed on the standard $xy$-plane), with domain $[0,\infty)$ and $y$-intercept (0,5), and response contains work showing the use of a table. |   $x$ \| $y$: 0 \| 5; 1 \| 6; 2 \| |
| **Level 3a response** | The teacher's graph has the same shape as a root function (such as $y = \sqrt{\lvert x \rvert} + 5$) graphed on the standard $xy$-plane with domain $(-\infty,\infty)$ and $y$-intercept (0,5). |   |
| **Level 3b response** | The teacher's graph has the same shape as the graph of $x = (y - 5)^2$ (graphed on the standard $xy$-plane) with domain $[0,\infty)$ and vertex (0,5). |   |
| **Level 2 response** | The teacher's graph is an upward-facing parabola with domain $(-\infty,\infty)$ or $[0, \infty)$ and vertex (0,5). |   |
| **Level 1 response** | The teacher graphed a collection of at least three points without connecting them. | |
| **Level 0 response** | *Any* of the following:<br>- The response does not fit a higher level.<br>- The scorer cannot interpret the response.<br>- The response consists of scratch work with no clear indication of a final answer.<br>- The response does not address the prompt; that is, the response is off-topic (see Purpose and Rationale).<br>- The page contains no work, but does contain at least one mark to suggest that the teacher saw this item. | |
| **IDK response** | The response consists only of "I don't know", or something equivalent that suggests that the teacher is unsure of how to respond. If the teacher stated uncertainty and gave an additional response, score the response ignoring the uncertainty. | |
| **X response** | The page is completely blank. | |