9.3 Undoing the Product Rule

In 9.2, we investigated how to find closed form antiderivatives of functions that fit the pattern of a Chain Rule derivative. In this section we explore a technique for finding antiderivatives based on the Product Rule for finding derivatives.

The Importance of Undoing the *Chain* Rule

Before developing this new idea, we should highlight the importance of Undoing the Chain rule from 9.2 because of how commonly it applies. You will likely have need for this technique more than any other in finding closed form antiderivatives. Furthermore, applying Undoing the Chain rule is often necessary in combination with or as part of the other techniques we will develop in this chapter. The bottom line is this: when it comes to antiderivatives, Undoing the Chain rule is everywhere. Whenever posed with finding closed form accumulation functions, you should be continually on the look-out for rate functions that require this technique.

When it comes to Undoing the Chain rule being involved with other antiderivative techniques, Undoing the Product Rule is no exception. We'll see this in the second half of this section.

The Idea of Undoing the Product Rule

Recall that the Product Rule determines the derivative of a function that is the product of other functions, i.e. a function that can be expressed as $f(x)g(x)$. The Product Rule can be expressed compactly as:

$$(f\,g)' \quad = \quad f\,g' \;+\; g\,f'$$

In other words, the derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.

Representing the Product Rule as $f\,g \overset{Derivative}{\rightarrow} f\,g' + g\,f'$, you might expect that Undoing the Product Rule could be represented this way $f\,g' + g\,f' \overset{Aniti-Derivative}{\rightarrow} f\,g$. But this is not the case for two reasons. First, rate functions rarely appear in the very specific form $f\,g' + g\,f'$, and second, whenever finding antiderivatives of rate functions that are a sum of terms, normally the antiderivative of each term is determined separately, and then added to express the antiderivative of the original rate function.

Instead, to show the idea of Undoing the Product Rule, let's begin again with the Product Rule:

$$(f\,g)' \quad = \quad f\,g' \;+\; g\,f'$$

Taking the antiderivative of both sides yields the following:

$$f\,g \quad = \quad \int f\,g'dx \; + \; \int g\,f'dx$$

The Undoing the Product rule technique is based upon this relationship and can be used when both of the following conditions are met:

1) the given rate function has the form $f\,g'$, but to find its antiderivative, neither an elementary rule nor Undoing the Chain Rule apply, and
2) for the associated rate function $g\,f'$, an elementary technique or Undoing the Chain rule *does* apply to find its antiderivative

This is summarized here:

$$f\,g \quad = \quad \int f\,g'\,dx \; + \; \int g\,f'\,dx$$

Given integral: anti-derivative desired, but unattainable by known techniques

Anti-derivative attainable

Thus, finding the desired antiderivative means determining $g\,f'$ instead. Once the antiderivative of $g\,f'$ is found, along with $f\,g$, the desired antiderivative can be solved indirectly by expressing it as:

$$\int f\,g'dx \quad = \quad f\,g \; - \; \int g\,f'dx$$

An Exploratory Example

Find the principal antiderivative of $x\cos x$.

Solution

A check for using Undoing the Chain Rule shows it does not apply. (Note that it would apply if the function instead were $x\cos x^2$.)

Considering Undoing the Product Rule, the given rate function $x\cos x$ has the role of $f\,g'$:

$$f\,g = \underbrace{\int f\,g'}_{\substack{\uparrow \\ x\cos x}}\,dx + \int g\,f'dx$$

This means that x or $\cos x$ need to be assigned to g' and f, creating two possible assignments. We will explore both to see which choice leads to finding the antiderivative of $x\cos x$.

Assignment 1) $x \to g'$ and $\cos x \to f$

$$x \to g' \quad \text{and} \quad \cos x \to f$$
$$\downarrow$$
$$f\,g = (\cos x)\frac{x^2}{2}$$
$$\downarrow$$
$$(f\,g)' = (\cos x)x + \frac{x^2}{2}(-\sin x)$$
$$\downarrow$$
$$f\,g = \underbrace{\int x\cos x\,dx}_{\text{Desired Anti-Deriv}} + \underbrace{\int \frac{x^2}{2}(-\sin x)\,dx}_{\text{Easier Anti-Deriv (?)}}$$

In the spirit of Undoing the Product Rule, we'd be able to determine the antiderivative of this alternative function, $\frac{x^2}{2}(-\sin x)$, more readily than the original function $x\cos x$. We'll investigate the other assignment for g' and f, and compare the results.

Assignment 2) $\cos x \to g'$ and $x \to f$:

$$\cos x \to g' \quad \text{and} \quad x \to f$$
$$\downarrow$$
$$f\,g = x(\sin x)$$
$$\downarrow$$
$$(f\,g)' = x\,(\cos x) + (\sin x)\cdot 1$$
$$\downarrow$$
$$f\,g = \underbrace{\int x\cos x\,dx}_{\text{Desired Anti-Deriv}} + \underbrace{\int \sin x\,dx}_{\text{Easier Anti-Deriv (?)}}$$

With assignment 2), the alternate antiderivative needed is that of $\sin x$, which has an immediate solution, whereas the antiderivative of $\frac{x^2}{2}(-\sin x)$ from assignment 1 above does not. Assignment 2) is the clear path to determining $\int x\cos x\,dx$.

Finishing the solution from where we left off in Assignment 2)…

$$\begin{array}{ccccc} f\,g & = & \int x\cos x\,dx & + & \int \sin x\,dx \\ & & \downarrow & & \\ \int x\cos x\,dx & = & f\,g & - & \int \sin x\,dx \\ & & \downarrow & & \\ & = & x\sin x & - & (-\cos x) \end{array}$$

Thus, $x\sin x + \cos x$ is the principal antiderivative of $x\cos x$.

Reflecting on the $x\cos x$ Example

Like $x\cos x$, rate functions with the general form $x^n k(x)$, where k is a sinusoidal or exponential function, are often good candidates for Undoing the Product Rule. Here $x^n k(x)$ takes on the role of $f\,g'$. Notice that in the $x\cos x$ example, the assignment of $\cos x \to g'$ and $x \to f$ is successful because the power rule for finding f' means its exponent is one less than f. The lower degree of f' results in the rate function $g\,f'$ having a simpler antiderivative: $g\,f' = (\sin x)x^0 = \sin x$.

This observation informs us of how to make the assignment of g' and f in the case of $x^n k(x)$. While not a hard and fast rule, assigning $k(x) \to g'$ and $x^n \to f$ is usually the path to the solution; knowing this helps to avoid having to try both assignments in future problems.

Example Find the principal antiderivative of $x^2 e^x$.

Solution

First check for Undoing the Chain Rule; it does not apply. (Observe that Undoing the Chain Rule would apply if the function were $x^2 e^{x^3}$ instead.)

We note that this function is a good candidate for Undoing the Product Rule since it has the form $x^n k(x)$. From the previous example, the correct assignment is likely $e^x \to g'$ and $x^2 \to f$:

$$
\begin{gathered}
e^x \rightarrow g' \quad \text{and} \quad x^2 \rightarrow f \\
\downarrow \\
f\,g = x^2 e^x \\
\downarrow \\
(f\,g)' \;=\; x^2 e^x \quad + \quad 2x\,e^x \\
\downarrow \\
f\,g \;=\; \underbrace{\int x^2 e^x \, dx}_{\text{Desired Anti-Deriv}} \quad + \quad \underbrace{\int 2x\,e^x \, dx}_{\text{Easier Anti-Deriv (?)}}
\end{gathered}
$$

Using this method, we anticipate that the alternate antiderivative $\int 2x\,e^x\,dx$ is simpler than the original, $\int x^2 e^x\,dx$. Is it? Yes and no: the degree of x in has been reduced from 2 to 1, but $\int 2x\,e^x\,dx$ does not have an immediate solution.

You may be tempted to conclude our initial assignment of g' and f was incorrect. But swapping them results in needing to find $\int \frac{1}{3}x^3 e^x\,dx$ instead of $\int 2x\,e^x\,dx$. This assignment is even worse!

Look again at $\int 2x\,e^x\,dx$. Undoing the Chain Rule does not apply, but is there another method that could work? Undoing the Product Rule! $2x\,e^x$ has the form $x^n k(x)$, so we can use this technique a second time within the first to complete the solution.

We will not show the work here, but you should verify with Undoing the Product Rule that the principal antiderivative of $2x\,e^x$ is $2x\,e^x - 2e^x$. Thus,

$$
\begin{aligned}
f\,g \;&=\; \underbrace{\int x^2 e^x \, dx}_{\text{Desired Anti-Deriv}} \quad + \quad \underbrace{\int 2x\,e^x \, dx}_{\text{Easier Anti-Deriv}} \\
&\downarrow \\
\int x^2 e^x\,dx \;&=\; f\,g - \int 2x\,e^x\,dx \\
&\downarrow \\
\int x^2 e^x\,dx \;&=\; x^2 e^x - \left(2xe^x - 2e^x\right) \\
&\downarrow \\
\int x^2 e^x\,dx \;&=\; x^2 e^x - 2xe^x + 2e^x
\end{aligned}
$$

Sub-problem also uses Undoing the Product Rule

Note that because of the factor x^2, the solution required using the technique twice. We can generalize that finding a closed form of $\int x^n e^x \, dx$ will require applying the Undoing the Product Rule technique n times – each use is nested in the solution of the previous use.

Reflection Question 9.3.1 Smoke from a forest fire accumulates at a rate of $m^2 e^m$ ft^3 / minute, where m is the number of minutes after 4 p.m. Find the volume of smoke accumulated between 4:03 p.m. and 4:06 p.m. Write and use an accumulation function to answer this question. (Hint: don't repeat any work that's already been done in this section!)

Example Find the principal derivative of $\ln x$.

Soloution

The function $\ln x$ does not fit the form of Undoing the Chain Rule, nor is there an elementary function that has $\ln x$ as its derivative. Applying Undoing the Product Rule means that $\ln x$ has the role of $f\ g'$, so consider it in the form $1 \cdot \ln x$. Finding $f\,g$ requires finding the antiderivative of g', so assigning $\ln x \to g'$ will not work because determining g is the very problem we started with: the antiderivative of $\ln x$! Thus, the assignment of g' and f is determined, and the solution is straightforward:

$$1 \to g' \quad \text{and} \quad \ln x \to f$$

$$\downarrow$$

$$f\ g = (\ln x)x$$

$$\downarrow$$

$$(f\ g)' = \ln x + \frac{1}{x}x$$

$$\downarrow$$

$$f\ g = \underbrace{\int \ln x \, dx}_{\text{Desired Anti-Deriv}} + \underbrace{\int 1 \, dx}_{\text{Easier Anti-Deriv!}}$$

And so

$$\begin{aligned} \int \ln x \, dx &= f\ g - \int 1 \, dx \\ &= (\ln x)x - x \end{aligned}$$

Reflection Question 9.3.2 We generalized earlier that finding a closed form of $\int x^n e^x\,dx$ requires n iterations of applying the Undoing the Product Rule technique. How many iterations are required to find a closed form for $\int x^5 \ln x\,dx$? Justify (or revise) your answer by beginning the process of finding the closed form antiderivative, and completing enough work to make a definitive conclusion. Explain and summarize your findings.

Example Find the principal antiderivative of $\dfrac{x^3}{\sqrt{4+x^2}}$.

Solution

Check for Undoing the Chain Rule first; it does not apply. (Two possible variations of this function that would make it eligible for Undoing Chain are $\dfrac{x^2}{\sqrt{4+x^3}}$, and $\dfrac{x}{\sqrt{4+x^2}}$.)

Applying Undoing the Product Rule, we first try the assignment $(4+x^2)^{-1/2} \to g'$ and $x^3 \to f$. Here, finding the antiderivative of g' is not elementary, indicating to swap the terms and try the other assignment:

$$x^3 \to g' \quad \text{and} \quad (4+x^2)^{-1/2} \to f$$

$$\downarrow$$

$$f\,g = (4+x^2)^{-1/2} \cdot \frac{1}{4}x^4$$

$$\downarrow$$

$$(f\,g)' = (4+x^2)^{-1/2}x^3 + \frac{1}{4}x^4\left(-\frac{1}{2}(4+x^2)^{-3/2}(2x)\right)$$

$$\downarrow$$

$$f\,g = \underbrace{\int (4+x^2)^{-1/2}x^3\,dx}_{\text{Desierd Anti-Deriv}} + \underbrace{\int\left(-\frac{1}{4}x^5(4+x^2)^{-3/2}\right)dx}_{\text{Easier Anti-Deriv (?) No!}}$$

Clearly, the alternate antiderivative at right is not easier, and it seems that both assignments of f and g' lead to a dead end. We need some ingenuity here.

Consider again the first assignment: $(4+x^2)^{-1/2} \to g'$ and $x^3 \to f$. We could find g by Undoing the Chain Rule if we had a factor of x along with $(4+x^2)^{-1/2}$ for g'. Can we somehow engineer that? Noticing the x^3 that's assigned to f, we can steal one of those x's! In other words, make the assignment $x(4+x^2)^{-1/2} \to g'$ and $x^2 \to f$. Undoing the Chain Rule now applies to g', and we solve that sub-problem first.

Recall, Undoing the Chain Rule means our rate function $x(4+x^2)^{-1/2}$ has the form $k\ G'(F(x))F'(x)$. The composite function $(4+x^2)^{-1/2}$ has the role of $G'(F(x))$.

The first attempt consists of the antiderivative of "something to the -1/2" leaving the argument unchanged. So the first attempt is $2(4+x^2)^{1/2}$.

To check and find a possible constant multiple needed, take the derivative of the first attempt:

$2\cdot\frac{1}{2}(4+x^2)^{-1/2}\cdot(2x)=2x(4+x^2)^{-1/2}$. This is 2 times as much as the original function, so we multiply the first attempt by 1/2 to get the finalized antiderivative:

$$\frac{1}{2}\cdot 2(4+x^2)^{1/2} \quad = \quad (4+x^2)^{1/2}$$

This function is *g* in the larger scheme of Undoing the Product Rule, so we're ready to proceed:

$$x(4+x^2)^{-1/2} \to g' \quad \text{and} \quad x^2 \to f$$

$$\downarrow$$

$$f\ g = x^2\cdot(4+x^2)^{1/2}$$

$$\downarrow$$

$$(f\ g)' \; = \; x^2\cdot\frac{1}{2}(4+x^2)^{-1/2}\cdot 2x \quad + \quad (4+x^2)^{1/2}\cdot 2x$$

$$\downarrow$$

$$f\ g \; = \; \underbrace{\int x^3(4+x^2)^{-1/2}\,dx}_{\text{Desierd Anti-Deriv}} \; + \; \underbrace{\int (4+x^2)^{1/2}\cdot 2x\,dx}_{\text{Easier Anti-Deriv (?)}}$$

Is the antiderivative at right attainable? You should recognize it as another Undoing the Chain Rule problem, so yes! The factor $2x$ is a constant multiple of the derivative of $4+x^2$. As is the goal with such antiderivatives stated in Exercise Set 9.2 #3, we solve this one all at once, by asking "what function's derivative is $(4+x^2)^{1/2}\cdot 2x$?" The Chain Rule applied to $\frac{2}{3}(4+x^2)^{3/2}$ produces this given function. Thus,

$$f\,g \;=\; \int x^3(4+x^2)^{-1/2}\,dx \;+\; \int (4+x^2)^{1/2}\cdot 2x\,dx$$

$$\downarrow$$

$$\int x^3(4+x^2)^{-1/2}\,dx \;=\; f\,g \;-\; \int (4+x^2)^{1/2}\cdot 2x\,dx$$

$$\downarrow$$

$$\int x^3(4+x^2)^{-1/2}\,dx \;=\; x^2(4+x^2)^{1/2} \;-\; \frac{2}{3}(4+x^2)^{3/2}$$

Common Forms for Undoing the Product Rule

As our toolbox of techniques for finding closed-form antiderivatives grows, we need to gain skill at recognizing which tool applies in a given situation.

Remember to always check first for Undoing the Chain Rule, since it's the most prevalent technique.

If Undoing the Chain Rule does not apply, check to see if your function is from this list of these common forms for Undoing the Product Rule.

Undoing the Product Rule Common Forms (*a* and *b* are non-zero constants)

$\int ax^n e^{bx}\;dx$

$\int ax^n \sin bx\;dx$ (or cos *bx*)

$\int \ln x\;dx$

$\int ax^n \ln x\;dx$

Inverse Trig Functions: $\sin^{-1} x,\ \cos^{-1} x,\ \tan^{-1} x$

If your rate function has one of the forms above, you can and should use Undoing the Product Rule. But Undoing the Product Rule also works in other situations as well, knowing if it does or doesn't work in a particular case may require trial and error.

Reflection Question 9.3.3 How could Undoing the Product Rule be used for finding the closed form antiderivative of $\tan^{-1} x$ *? Determine the assignment of* g' *and f that would make this possible. In the solution, what technique is used in finding the antiderivative of the alternate / easier function?*

Exercise Set 9.3

1) Find the principal closed-form anti