9.4 Antiderivatives of Expressions of Trigonometric Functions

In this chapter, we are assembling a tool box of techniques for finding closed-form antiderivatives of rate of change functions. Before adding more tools, let's review some essential ideas related to this endeavor:

Reflection Question 9.4.1 What kind of function is the antiderivative of a rate of change function? Don't just state a word or phrase; explain what it is.

Reflection Question 9.4.2 Our goal in this chapter is finding functions in "closed form." Explain what we mean by functions expressed in "open form" and "closed form," and discuss the benefits of each.

Trigonometric Functions

If you know well the basic derivative rules for trigonometric functions, then determining the antiderivatives in the reflection below should be quick and easy.

Reflection Question 9.4.3 Find the principal antiderivative of each. Watch yours signs!

a) $\int \sin x\, dx$ b) $\int \cos x\, dx$ c) $\int \sec^2 x\, dx$ d) $\int \sec x \tan x\, dx$

e) $\int \csc^2 x\, dx$ f) $\int \csc x \cot x\, dx$

The antiderivatives in this section are of course more complex and difficult, but having these basic antiderivative facts at the ready will be crucial when it comes to solving the harder ones.

An Example that Models the Strategy

As a basic example of the general strategy we'll develop in this section, let's find the principal antiderivative of $\tan x$. First, rewrite the function using the basic *identity* for tangent:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = \int (\sin x)(\cos x)^{-1}\, dx$$

Then check first for Undoing the Chain Rule: does $(\sin x)(\cos x)^{-1}$ have the form $k\, g'(f(x))\, f'(x)$? Yes, since $\sin x$ is a constant times the derivative of the argument of $(\cos x)^{-1}$.

Thus, $(\cos x)^{-1}$ has the role of $g'(f(x))$, and the first attempt is $\ln|\cos x|$.

Checking this via the derivative, we get $\dfrac{-\sin x}{\cos x} = -\tan x$ so we need the opposite of the first attempt.

The principal antiderivative of $\tan x$ is therefore $-\ln|\cos x|$.

Strategy for Antiderivatives of Trig Functions

In the $\tan x$ example, we followed this approach which is typical for expressions comprised of trig functions:

1) Use trig identities and / or algebra to rewrite the given rate function in a form that either
   a) has an immediate solution or
   b) is eligible for Undoing the Chain Rule

2) Find the anti-derivative according to a) or b) above, whatever applies

In summary, to solve these antiderivatives:

Use identities and / or algebra to set the stage for Undoing the Chain Rule (or an elementary solution)

Trig Identities

Since this strategy is dependent on using trig identities, we'll need a list of the most important ones to draw from:

Basic Definitions

$\tan x = \dfrac{\sin x}{\cos x}$ $\qquad \cot x = \dfrac{\cos x}{\sin x} = \dfrac{1}{\tan x}$ $\qquad \sec x = \dfrac{1}{\cos x}$ $\qquad \csc x = \dfrac{1}{\sin x}$

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$

Double Angle Identities

$$\sin 2x = 2\sin x \cos x \qquad \cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$
$$= 2\cos^2 x - 1$$

Half-Angle Identities

$$\sin^2 x = \frac{1}{2}\left(1 - \cos 2x\right) \qquad \cos^2 x = \frac{1}{2}\left(1 + \cos 2x\right)$$

Sum Identities

$$\sin(x + y) = \sin x \cos y + \sin y \cos x$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

Helpful Tips & Examples

Again, the overall approach in finding these antiderivatives is to rearrange the rate function in order to set the stage for Undoing the Chain Rule, or to get something easy.

But there are a few helpful tips we should mention before doing some examples:

TIPS

1) Sometimes, for functions that involve tangent, cotangent, secant and cosecant, it helps to first rewrite the function entirely in terms of sine and cosine using the basic definition identities. This does NOT always apply; knowing in a particular case may require trial and error.
2) In changing the form of the rate function, work to assure that all factors and terms have the same argument. For example, $(\cos 2x)(\sin x)$ has different arguments, x and $2x$, but its alternate form $(\cos^2 x - \sin^2 x)(\sin x)$ has a universal argument, x